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Quasiclassical corrections in functional integral method and applications

*Półklasyczne poprawki w metodzie całek funkcjonalnych
i zastosowania*

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To my supervisor Sergey Leble
for unwavering support and immense patience

List of my publications:

1. G. Kwiatkowski, S. Leble, *Quantum corrections to SG equation solutions and applications*, Physics Letters A **376**, 8-9, pp. 991-995 (2012)

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3. G. Kwiatkowski, S. Leble, *Quantum corrections to ϕ^4 model solutions and applications to Heisenberg chain dynamics*, Central European Journal of Physics **11**, 7, pp. 887-893 (2013)

4. G. Kwiatkowski, S. Leble, *Quantum corrections to quasi-periodic solution of Sine-Gordon model and periodic solution of ϕ^4 model*, Journal of Physics: Conference Series **482**, 012023 (2014)

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Chapter 1

Introduction

1.1 Historical overview

1.1.1 Solitons

Nowadays solitons are one of the most commonly studied nonlinear fields. They are roughly defined as pulses, which retain their exact shape despite dispersion and collisions. Such quality is of high interest in communication technology as well as elementary particle models. Solitary waves were first documented and studied experimentally by Scott-Russell in 1844 [1]. First theoretical explanation came almost 30 years later in works of Boussinesq [2, 3] and Rayleigh [4] on shallow water wave equations and finally in 1895 Korteweg and de Vries published their most renown paper [5], in which they found analytic fixed-form solution of

$$\frac{\partial \phi}{\partial t} + \frac{\partial^3 \phi}{\partial x^3} + 6\phi \frac{\partial \phi}{\partial x} = 0, \tag{1.1.1}$$

which they derived independently of similar works by Boussinesq from 1877 [6]. The aforementioned solution

$$\phi(t, x) = 2\kappa^2 \text{sech}^2(4\kappa^3 t - \kappa x) \tag{1.1.2}$$

(with κ as a free parameter) does not disperse like a typical linear wave packet. In fact, it keeps its shape even after colliding with other waves. Aforementioned solution matched well with the general shape and properties of solitary waves studied by Scott-Russell. It is of note, that it could not be approximated using perturbation techniques. Even though nonlinear differential equations were first formally studied in differential geometry in the context of Gauss-Peterson-Codazzi equations, which are central to the problem of embedding of submanifolds, the results of Korteweg and de Vries were the first non-trivial nonlinear solutions of a field equation. For instance, the so called Sine-Gordon equation

$$\frac{\partial^2 \phi}{\partial u_1 \partial u_2} = m^2 \sin \phi \tag{1.1.3}$$
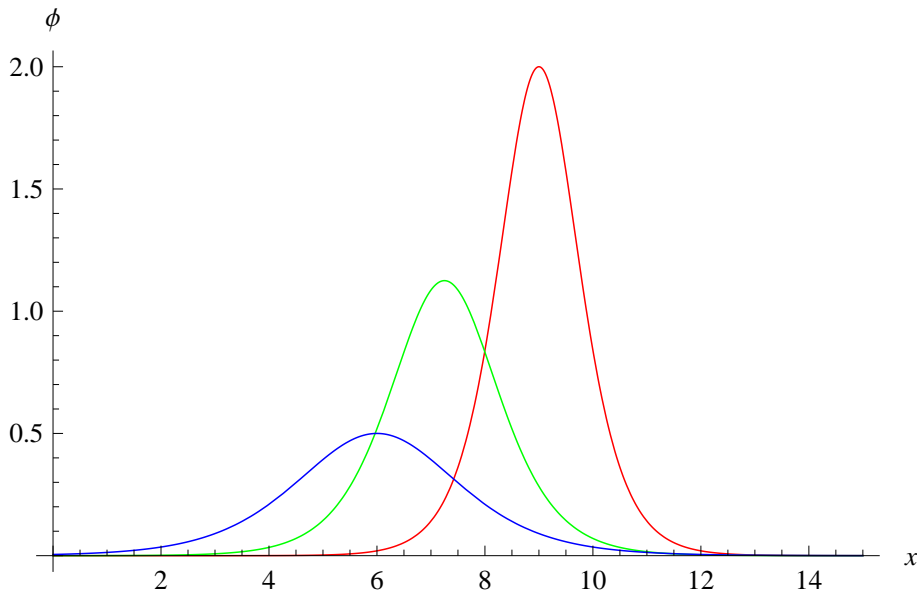
Figure 1.1: KdV solitary waves for $\kappa \in \{\frac{1}{2}, \frac{3}{4}, 1\}$ (from left to right).

(in light cone coordinates) was first used in the study of pseudospheres back in 1862 [7]. Yet, it wasn't until 1939 for it to find a use in physics as a continuum approximation of the Frenkel-Kontorova model of crystalline structures [8] and it took another 11 years for soliton (a term coined by Zabusky and Kruskal in their publication on KdV solitary waves collisions in 1965 [9]) solutions to be discovered by Kochendörfer and Seeger [10, 11]

$$\phi(u_1, u_2) = 2 \arcsin\{\operatorname{tgh}[m(a_1 u_1 + a_2 u_2)]\} + \pi \tag{1.1.4}$$

(with a_1 and a_2 as arbitrary parameters with condition $a_1 a_2 = 1$) or

$$\phi(t, x) = 2 \arcsin\left[\operatorname{tgh}\left(\frac{m}{\sqrt{1-v^2}}x - \frac{mv}{\sqrt{1-v^2}}t\right)\right] + \pi \tag{1.1.5}$$

in normal Cartesian coordinates. The most important feature of this solution is that it can be stationary or propagate with an arbitrary speed lower than that of linear waves of the given system. In essence, it behaves as a particle with a non-zero mass. The notion is further reinforced by the fact, that there is a conserved quantity connected to the total number of solitons. Kochendörfer's and Seeger's results included solutions containing arbitrary number of solitons (or strictly speaking kinks) and so-called breathers (a localized oscillating pulse, which can be seen as a bound state of a kink-antikink pair). It is important to stress, that along with low amplitude waves this is a complete set of solutions for Sine-Gordon equation, proving which is a rare achievement in case of a nonlinear system (see [111]). Moreover, Faddeev and Takhtadzhan were able to transform the Hamiltonian structure determined by Poisson bracket

$$\{\pi(x), \phi(y)\} = \delta(x - y) \qquad \pi(x) = \frac{1}{\gamma}\frac{\partial \phi}{\partial t}(x) \tag{1.1.6}$$
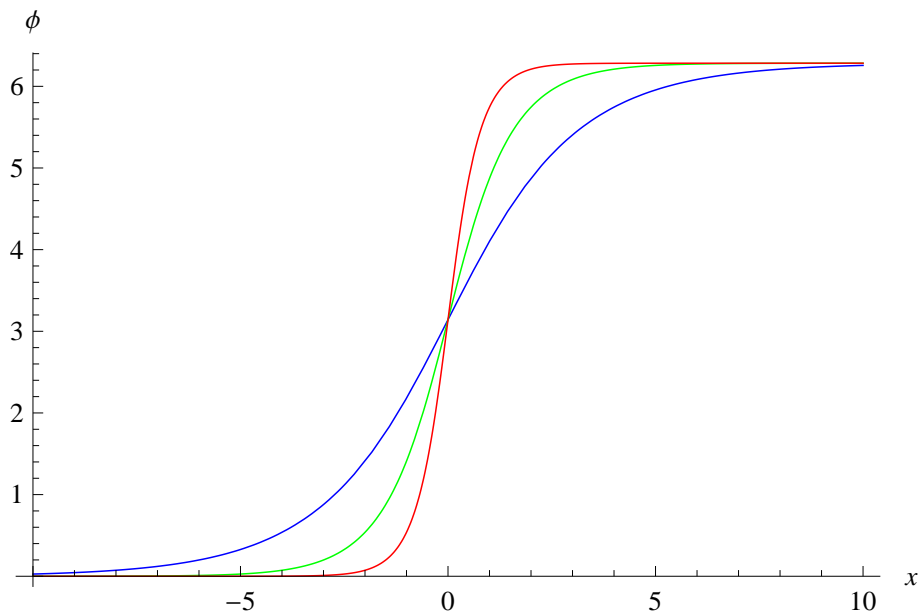
Figure 1.2: Stationary Sine-Gordon kink in Cartesian coordinates for $m = 0.5$ (blue), $m = 1$ (green), $m = 2$ (red).

(with γ as a coupling constant) into a set of three independent subsystems [111]

1. $0 \leqslant \rho(p) < \infty, \quad 0 \leqslant \chi(p) < 2\pi, \quad \{\rho(p_1), \chi(p_2)\} = \delta(p_1 - p_2),$

2. $-\infty < p_a < \infty, \quad -\infty < q_a < \infty, \quad \{p_{a_1}, q_{a_2}\} = \delta_{a_1 a_2}, \quad a \in \{1, \dots, A\},$

3. $-\infty < \xi_b < \infty, \quad -\infty < \eta_b < \infty, \quad 0 \leqslant \alpha_b < 2\pi, \quad 0 \leqslant \beta_b < \frac{8\pi}{\gamma},$
   $\{\xi_{b_1}, \eta_{b_2}\} = \delta_{b_1 b_2}, \quad \{\alpha_{b_1}, \beta_{b_2}\} = \delta_{b_1 b_2}, \quad b \in \{1, \dots, B\}$

describing regular waves, single solitons and breathers respectively with A and B as arbitrary integers denoting the number of particles of a given type. Sine-Gordon equation has proven to be substantial in describing dynamics of crystal defects including adsorbed atomic layers [12] and surface reconstruction [13]. Due to the integrability of Sine-Gordon equation and richness of possible solutions, it found use in a variety of research fields ranging from spin structure dynamics [14, 15] to elementary particle physics [16, 17, 18]. For a more detailed description of the Frenkel-Kontorova and Sine-Gordon systems in the context of crystal and magnetic structures see [19]. Another widely used nonlinear equation (usually denoted as ϕ^4 model)

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + C_1 \phi - C_2 \phi^3 = 0 \tag{1.1.7}$$

was introduced by Landau in his paper on phase transitions [20]. Considering it is the simplest wave equation with a polynomial nonlinearity, it can be found in many works using perturbative methods to deal with nonlinear systems. Most importantly, it contains a stable kink solution for $C_1 C_2 > 0$, which behaves as a relativistic particle of non-zero mass just as the Sine-Gordon kink, yet in the ϕ^4 model, there

can never exist two kinks or antikinks in a row, since there are only two minima in the potential. A similar potential emerges naturally in the mean field approximation
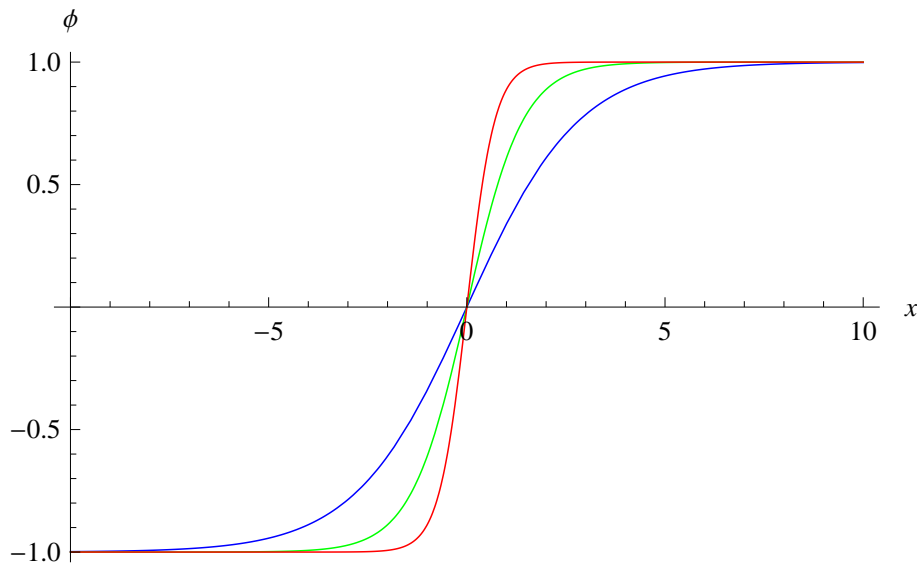
Figure 1.3: Stationary ϕ^4 kink in Cartesian coordinates for $m = 0.5$ (blue), $m = 1$ (green), $m = 2$ (red) with $V = \sqrt{2}$.

of multiparticle problems in quantum mechanics leading to the well known nonlinear Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \psi}{\partial x^2} + \kappa|\psi|^2\psi, \tag{1.1.8}$$

which can be applied to nonlinear optical fibres, planar waveguides and Bose-Einstein condensation. It is of note, that whereas nonlinear Schrödinger equation is completely integrable [21] the ϕ^4 model is not integrable.

1.1.2 Quantization

Canonical quantization

The first general formulation of quantization process was given by Heisenberg in 1925 [22] and further refined in collaboration with Born and Jordan [23, 24]. The general idea is to describe observable quantities as time-dependent matrices indexed over all possible energy states of the system with the equivalents of position and momentum still fulfilling equations of Hamiltonian mechanics

$$\begin{cases} \frac{dX}{dt} &= \frac{\partial H}{\partial P} \\ \frac{dP}{dt} &= -\frac{\partial H}{\partial X} \end{cases} \tag{1.1.9}$$

with Hamiltonian H as in the classical system with the distinction, that X and P don't commute, which will have an impact on the results. In order to make the

solutions unequivocal, Heisenberg added a normalising condition on the commutator of X and P (as well as all other observables)

$$[X, P] = i\hbar I \qquad (1.1.10)$$

with I as identity matrix. Value of the constant on the right-hand side stems from the earlier findings on periodic motion in Bohr atom model as well as the study of quantum harmonic oscillator. It is important to remark, that due to (1.1.9) all commutation relations are direct analogs of Poisson bracket relations in classical mechanics, which is even more prominent, when one looks at the Heisenberg's equation of motion for observables

$$i\hbar \frac{\partial}{\partial t} O = [O, H] \qquad (1.1.11)$$

(with O as an arbitrary observable), which is a direct analog of the classic condition for conservation of a given quantity

$$\frac{\partial}{\partial t} O = \{H, O\}. \qquad (1.1.12)$$

It can be said, that the quantization process is the act of substitution of Poisson bracket with a different operation. This notion was further generalized by Moyal [25] and independently by Groenewold [26], who introduced a deformation of Poisson bracket (parametrised by $\hbar$), which also behaves as a deformed commutator for non-commuting quantities. This formalism is often called minimal or deformation quantization and is a useful tool in studying relations between classical and quantum systems. For instance, it was recently adopted by Błaszak and Domański [27, 28] and used to analyse separability of variables in quantum versions of certain integrable systems [29, 30]. As for the Heisenberg's picture, position matrix X contains information about available energy states and possible jumps between them, but it does not describe a specific state of the system anymore. For that, additional state vectors ϕ had to be introduced and average values of a given observable were obtained as (here for example the position)

$$\sum_{n,k} \overline{\phi}_n X_{nk}(t) \phi_k. \qquad (1.1.13)$$

In this sense ϕ_n is the probability amplitude of finding the system in state n. It is worth stressing, that in the Heisenberg picture, the state vectors are not dependent on time, since the time evolution of the system is contained in the observables. The situation can be reversed, if we solve the evolution equation (1.1.11)

$$O(t) = e^{\frac{it}{\hbar}H} O(0) e^{-\frac{it}{\hbar}H} \qquad (1.1.14)$$

(with the exponential function defined through Taylor expansion) and insert the solution into (1.1.13). We can rightfully assume, that the time evolution operator $e^{-\frac{it}{\hbar}H}$ works on the state vector ϕ instead of the observable matrix and define a time-dependent state vector

$$\phi(t) = e^{-\frac{it}{\hbar}H} \phi. \qquad (1.1.15)$$

This assumption is the basis of Schrödinger's series of papers [31, 32, 33, 34], where he treated observables as functional operators instead of using matrix representation. More precisely, he assumed a specific form of said operators

$$\begin{cases} \hat{x} &= \overrightarrow{x} \\ \hat{p} &= -i\hbar\nabla \\ \hat{E} &= i\hbar\frac{\partial}{\partial t} \end{cases} \tag{1.1.16}$$

and expanded the evolution equation in a Taylor series in t obtaining the well known differential equation

$$\hat{H}\psi(\overrightarrow{x}, t) = \hat{E}\psi(\overrightarrow{x}, t), \tag{1.1.17}$$

where the Hamiltonian operator $\hat{H}$ is obtained by a substitution of position and momentum by their operator forms in the classical formula. What is important, Schrödinger form of quantum mechanics preserves all commutation relations given by Heisenberg, yet it is strictly non-relativistic, whereas Heisenberg viewpoint can contain relativistic case as well as the non-relativistic one. This was the reason for Dirac to search for a different wave equation, which would be consistent with special relativity. He achieved his goal in 1928 [35] (here for a free particle)

$$i\hbar\gamma^{\mu}\partial_{\mu}\psi - mc\psi = 0 \tag{1.1.18}$$

here written down using the Einstein summation convention and with γ^{μ} as four-dimensional square matrices fulfilling anticommutation relation

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu,\nu} \tag{1.1.19}$$

with $g^{\mu,\nu}$ as the standard metric tensor. One of the implications was that the wave function was no longer a scalar and it implied a change in the definition of probability density. Even more important implication was the seemingly artificial negative energy spectrum mirroring the positive eigenvalues. Dirac proposed a conceptual solution to that problem [36] by postulating, that almost all negative energy states are already occupied. Going from this assumption he predicted, that holes (akin to those encountered in semiconductors) behave exactly like electrons with an opposite electrical charge. This concept also readily led him to discover the possibility of annihilation and creation processes. He did however make a false assumption, that the positively charged counterparts to electrons are protons - a mistake, that was however soon rectified by Weyl, who pointed out, that Dirac's hole would have to have the same mass as an electron. Thus a positron was the first particle predicted theoretically before it was ever observed, which happened in 1932 [37, 38]. Meanwhile Dirac has stated that relativistic quantum mechanics has to explicitly involve the field through which particles interact and are observed [39]. The resulting model for quantum electrodynamics was further studied by Fock and Podolsky [40] and quickly led to a joint paper with Dirac [41], where they have shown the equivalence of this new model to the earlier attempts on quantization of electromagnetic field by Heisenberg and Pauli [42, 43], which gives a full view of the so called old quantum

electrodynamics. The problem of infinite ground state energy due to an infinity of degrees of freedom was not satisfactorily solved until 1947, when Bethe proposed a first mathematically sound way of renormalization [44]. At the same time Feynman has sought for an entirely new way of quantization in hope to rewrite the quantum electrodynamics in a more universal and coherent way. His approach to quantization was formulated as an extrapolation of probability amplitude summation rules in 1948 [45] along with the notion, that lack of information about the quantum state has physical consequences. More precisely, he defined the wave function as a sum over all possible histories leading to the current state (a general idea, which was loosely mentioned in earlier papers by Dirac [39])

$$\psi(x,t) = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \prod_{k=-\infty}^{-1} e^{\frac{i}{\hbar}S(x_k, x_{k+1})} \prod_{k=-\infty}^{-1} \frac{dx_k}{A_k} \qquad (1.1.20)$$

with x_k as positions at times t_k, S as action integral over a classical path between events (t_k, x_k) and (t_{k+1}, x_{k+1}), $x_0 = x$, $t_0 = t$, ϵ as the partition width, $t_k \to -\infty$ as $k \to -\infty$ and A as a normalising factor. In this form it is not computable, but it naturally shows the way to calculate evolution of a known state

$$\psi(x,t) = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \psi(x_{-N_\epsilon}, t_{-N_\epsilon}) \prod_{k=-N_\epsilon}^{-1} e^{\frac{i}{\hbar}S(x_k, x_{k+1})} \prod_{k=-N_\epsilon}^{-1} \frac{dx_k}{A_k} \qquad (1.1.21)$$

with t_{-N_ϵ} as a fixed time previous to t and $N_\epsilon \to \infty$ as $\epsilon \to 0$. A year later Feynman published two papers concerning quantum electrodynamics [46, 47]. By combining propagator formalism with Stückelberg's idea of positrons as electrons travelling backward in time he obtained creation and annihilation processes as a type of scattering process in an unmodified Dirac system. The key was to change the typical condition on the propagator kernel (denoted with K)

$$\forall_{t' < t} K(q, q', t, t') = 0 \qquad (1.1.22)$$

(with primed coordinates as the final coordinates) with one based on the distinction between positive eigenvalues of the Hamiltonian (associated with electrons) and the negative ones (associated with positrons)

$$\begin{cases} \forall_{t' > t} K(q, q', t, t') = \sum_{E_n > 0} \phi_n(x', t') \overline{\phi}_n(x', t') e^{-iE_n(t'-t)} \\ \forall_{t' < t} K(q, q', t, t') = -\sum_{E_n < 0} \phi_n(x', t') \overline{\phi}_n(x', t') e^{-iE_n(t'-t)}. \end{cases} \qquad (1.1.23)$$

Within this formalism Feynman has described virtual particles as closed loops in space-time, which led to a relatively simple form of vacuum to vacuum amplitude (denoted as C_v)

$$C_v = e^{-L}, \qquad (1.1.24)$$

where L is the sum of amplitudes from all possible single virtual particle loops, which is represented by a sum over single loops containing specific number of scattering processes. In a subsequent paper [48] he incorporated photons into the model by

taking advantage of the ease of separation of subsystems in his method of quantization. While the problem of self interaction and vacuum polarization was discussed, there was no definitive answer on the proper renormalization thereof. Throughout following decades many new techniques were developed in order to cut the emerging infinities, yet there were few works, which would redefine the quantum field theory in a way, which would not lead to divergent results in the first place. An interesting approach to the problem was introduced by Czachor in 2000 [49], where he proposed to treat frequency as another observable of the quantum system. It allowed him to construct vacuum state by using a finite amount of oscillators with indefinite frequency instead of using an infinity of oscillators as in the canonical representation of quantum electrodynamics (one for each allowed frequency). An analogue treatment of fermionic fields can be found in [50] and a discussion of possible experimental differences between canonical quantum electrodynamics and the proposed reducible representation was presented in [51, 52, 53]. It is also important to mention the algebro geometric approach to quantization, which allows for the use of the whole range of topological tools in order to investigate properties of quantum systems. An intuitive description was given by Tyurin [54] both for Abelian and non-Abelian structures.

Semiclassical quantization

However, for the purpose of this thesis we will focus on semiclassical quantization methods, which as yet are only developed for the canonical quantization schemes. First semiclassical scheme was the renown WKB approximation [55, 56, 57] (sometimes referred to as Liouville-Green method, since the same ideas can be found in their works from the first half of XIX century), which is a specific type of perturbation calculus. The general idea is to substitute the solution of a differential equation with an exponent containing a power series (starting from the -1 power and up to infinity) in a specific infinitesimal parameter. When applied to Schrödinger equation, it usually takes a slightly modified form:

$$\psi(x,t) = e^{\frac{i}{\hbar}S(x,t)} \sum_{n=0}^{\infty} \hbar^n \psi_n(x,t) \tag{1.1.25}$$

(with x containing all spatial variables). Separation of the equation in respect to powers of $\hbar$ gives as a series of equations, which can be used to calculate arbitrarily many elements of this expansion starting from S (which turns out to solve equations for corresponding classical action). In 1931 Fock has published a book on the fundamentals of quantum mechanics [58], which presents the state of the research field in that time. In his study of canonical transformations of quantum theory and their connection to contact transformation of classical mechanics he derived an expression for evolution of ψ_0 in the WKB approximation through the determinant of a Hessian

of the classical action integral in respect to initial and final coordinates

$$\Psi_Q(q) = (2\pi\hbar)^{\frac{n}{2}} \sqrt{\det\left[\frac{\partial^2 S}{\partial q \partial Q}\right]} e^{\frac{i}{\hbar}S(x,t)} \tag{1.1.26}$$

(with n as the number of spatial dimensions). This expression is basically the semi-classical approximation of the propagation operator even if it was not identified as such by Fock at the time. Exactly the same result was (possibly independently) derived by Pauli in 1951 [59] from Feynman's formulation of quantum mechanics for small propagation times

$$K(q, q', t) = \frac{1}{\sqrt{2\pi i\hbar}^n} \sqrt{D} e^{\frac{i}{\hbar}S(q,q',t)} \tag{1.1.27}$$

with K as the propagator kernel, n as the number of degrees of freedom, S as the classical action and D as the determinant of the Hessian of S in respect to all components of q and q'. He has also shown, that this approximation solves Schrödinger equation with an error of order $\hbar^2$, which is natural considering the connection with WKB scheme. In 1961 Maslov used Taylor expansion of the action integral around a chosen classical path with fixed border conditions

$$\psi(x,t) \approx \int_{-\infty}^{\infty} \psi(x_0, t_0) e^{\frac{i}{\hbar}\left(S(\varphi) + \frac{1}{2}\frac{\partial S}{\partial w_j \partial w_k}(\varphi)w_j w_k\right)} dx_0 \prod_o dw_o \tag{1.1.28}$$

(with φ as the classical path between (x_0, t_0) and (x, t), and w_o as coefficients in a representation of paths in a given base of functions) to approximate the path integral through functional determinants [60, 61]

$$\psi(x,t) \approx \int_{-\infty}^{\infty} \psi(x_0, t_0) \frac{e^{\frac{i}{\hbar}S(\varphi)}}{\sqrt{\det[L]}} dx_0 \tag{1.1.29}$$

(operator L derived from $\frac{\partial S}{\partial w_j \partial w_k}(\varphi)$ as $\frac{i}{2\hbar}\frac{\partial S}{\partial w_j \partial w_k}(\varphi) = -\pi(\phi_j, L\phi_k)$ with ϕ_j as chosen base functions). Later it has proven important in expanding Feynman ideas to nonlinear field theories, since this approach to semiclassical limit will not require more than a single classical path as opposed to the method shown by Pauli [59]. It is important to stress, that Maslov's method is non-perturbative in nature as opposed to WKB approximation. Meanwhile Garrod reformulated path integrals in the language of Hamilton mechanics [62] as opposed to the notion of action integral used in the original works of Feynman

$$\psi(x_1, t_1) = \int_{-\infty}^{\infty} \int_{x_0}^{x_1} \psi(x_0, t_0) e^{2\pi i \int_{t_0}^{t_1} (p(t)\dot{x}(t) - H(x(t),p(t)))dt} d[x(t)]d[p(t)]dx_0. \tag{1.1.30}$$

This allowed him to construct canonical transformations, which are much simpler in path integral form in arbitrary coordinate system, since they are performed on the classical coordinates and therefore do not require explicit operator form of position and momentum. Building upon those results Gutzwiller was able to construct the

semiclassical approximation of propagator kernel in the energy-momentum representation [63, 64]

$$\widetilde{K}(p'', p', E) = -\frac{1}{2\pi\hbar^2} \sum_{classical\ paths} |D_T|^{\frac{1}{2}} e^{i\frac{T}{\hbar} + \phi\frac{\pi}{2}} \tag{1.1.31}$$

with p' as the initial and p'' as final momentum, E as energy, T as classical action in energy-momentum representation, ϕ as phase factor connected with caustics in the phase space (see [63, 64] for details) later on identified as the winding number of the classical orbit [65] and D_T is a determinant of the Hessian of the classical action T in respect to E and all components of p' and p''. Apart from reproducing well known exact energy levels of Coulomb potential (derived as simple poles of the propagation kernel in the E variable), Gutzwiller has also given an explicit form of propagation kernel for a general spherically symmetric potential with a simple pole at the centre assuming only, that it is monotonic in r as well as given a proper way of extracting energy levels from the propagator. In two subsequent papers [66, 67] Gutzwiller refined the notion of state density

$$N(E) \propto -E \int K(q, q, E)dq + \int_0^E K(q, q, E')dqdE' \tag{1.1.32}$$

as an attempt to expand semiclassical quantization beyond the cases for which we can perform complete variable separation.

Quantization of solitons

Methods shown in Gutzwiller's papers allow one to calculate energy levels of a given potential without any knowledge of the wave functions, which was later utilised by Daschen, Hasslacher and Neveu in their most renown series of papers from 1974 [16], where they succeeded in quantising a static kink solution of the ϕ^4 system (wave equation with potential $V(\phi) = \frac{\lambda}{4}\phi^4 - \frac{m}{2}\phi^2$) by finding the correction to kink's energy

$$\Delta E = m\left(\frac{1}{2\sqrt{6}} - \frac{3}{\pi\sqrt{2}}\right) \tag{1.1.33}$$

(with $\hbar = c = 1$) as well energies of its excited states in the weak coupling approximation. It is of note, that the renormalization consisted of a regular vacuum cut-off and an additional additive term introduced in order to eliminate logarithmic divergences. Shortly thereafter Korepin together with Faddeev obtained quantum corrections to energy of a static soliton of the Sine-Gordon system ($V(\phi) = m^2(1 - \cos\phi)$) using Maslov approach to semiclassical quantization [68]

$$\Delta E = \text{Tr}\left[\ln\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + V''(\phi_{cl})\right)\right] \tag{1.1.34}$$

(here before regularization with ϕ_{cl} as the classical field). In the end the result was

$$\Delta E = -\frac{m}{\pi}. \tag{1.1.35}$$

It is of note, that their method includes time-dependent part of the action integral explicitly and does not depend on harmonic expansion of the potential around the classical field. Furthermore, it shows that Maslov approximation translates well to multidimensional problems. In the following year Dashen, Hasslacher an Neveu quantized the Sine-Gordon kink using the same method as for the ϕ^4 model and obtained the same result [17]. Actual calculations in a general d-dimensional theory required another improvement in mathematical methods, which came in form of generalized zeta-function first used for the purpose of regularization by Dowker and Critchley [69] in the context of Klein Gordon equation in de Sitter space

$$\zeta(x'', x', s, m^2) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} e^{-im^2\tau} K(x'', x', \tau) \qquad (1.1.36)$$

with τ as proper time in relativistic sense and m^2 as mass in the Klein-Gordon equation. It was used to calculate the semiclassical correction to Lagrangian

$$\Delta\mathcal{L}(x) = \lim_{s \to 1} \frac{i\zeta(x, x, s-1, m^2)}{2(s-1)} \qquad (1.1.37)$$

with $\hbar = c = 1$ and before renormalization, which can be performed on the level of finite zeta-function. Shortly after that Hawking constructed a general method for quantising field theories on curved space-time manifolds [70] using the generalized zeta-function. In 1980 Ventura explicitly calculated energy corrections to ϕ^4 in $3+1$ dimensions [71] using the method given earlier by Daschen et al. [16] (assuming $V(\phi) = \frac{\lambda}{4}\phi^4 - \frac{m^2}{2}\phi^2 + \frac{m^4}{4\lambda}$ and $\hbar = c = 1$)

$$\Delta E = \frac{m^3}{24\pi} \left(\frac{9\sqrt{2}}{\pi} - \sqrt{\frac{3}{2}} \right). \qquad (1.1.38)$$

Finally Konoplich was able to perform quantization of ϕ^4 kink solution in a general d-dimensional euclidean space [72] (assuming $V(\phi) = \frac{g}{4}\phi^4 - \frac{m^2}{2}\phi^2$ and $\hbar = c = 1$)

$$\Delta E_{1+1} = \frac{m}{\pi\sqrt{2}} \left[\frac{\pi}{2\sqrt{3}} - 3 - 3\ln\left(\frac{M}{m\sqrt{2}}\right) \right], \qquad (1.1.39)$$

$$\Delta E_{2+1} = \frac{3m^2}{4\pi} \left[1 + \frac{1}{4}\ln(3) \right], \qquad (1.1.40)$$

$$\Delta E_{3+1} = \frac{3m^3}{4\sqrt{2}\pi^2} \left[1 - \frac{\pi}{6\sqrt{3}} + \ln\left(\frac{M}{m\sqrt{2}}\right) \right]. \qquad (1.1.41)$$

Most notable in his work is the inclusion of the so called mass scale M introduced to cut logarithmic divergences, which is not an arbitrary parameter despite such an assertion in the publication. It is important to note, that the results are qualitatively dependent on the number of dimensions accounted for, even if the classical solution is not. Another important aspect of the method, which will not be elaborated on in this thesis, is that a very similar formalism can be used in thermodynamics, which was shown by Konoplich in [73]. A throughout explanation of Dashen's approach as

well as S-matrix formalism in the context of solitons can be found in the book by Rajaraman [74] (see also [75]). Later publications refined the mathematical methods, yet there was little progress in quantising nonlinear fields other than single solitons or kinks. Of note would be the development of direct mode summation by Bordag [76], which is a very general method suitable for localized potentials in $1+1$ and $3+1$ dimensions. Another important progress was made by Kirsten in collaboration with McKane [77] and Loya [78], when they proposed a method of constructing zeta-function by contour integrals without explicit solutions to the spectral problem. Namely, if the eigenvalue problem can be rewritten in a form

$$F(\lambda) = 0, \tag{1.1.42}$$

where F is a known function, then

$$\zeta(s) = \frac{1}{2\pi i} \int_\gamma d\lambda \lambda^{-s} \frac{d}{d\lambda} \ln(F(\lambda)), \tag{1.1.43}$$

where γ is a counterclockwise curve enclosing all the zeros of F. In recent years Pawellek has made a major breakthrough by quantising periodic solutions of Sine-Gordon and ϕ^4 systems [79, 80] using the methods developed by Kirsten. As important as those results are, they lack an explicit analytic form and they don't account for rest variables.

This survey would not be complete without the mention of quantum inverse scattering method. For those nonlinear systems, for which there exists a canonical transformation of coordinates into a fully linear Hamilton system (complete integrable systems, see [111] for an example) it is possible to go through with a canonical quantization procedure (substitution of Poisson bracket structure with commutators). In 1979 Sklyanin, Takhtadzhyan and Faddeev used such a representation to fully quantize the Sine-Gordon system [81]. It took another 10 years to quantize the Sinh-Gordon model [82]. The main limitation of quantum inverse scattering method is the requirement of complete integrability imposed on the classical system and it was only ever employed in case of $1+1$ dimensions (due to difficulties with studying multidimensional nonlinear classical systems) with infinite spatial domain (due to limitations of scattering methods). A comprehensive explanation of the method can be found in [83] with an application to Heisenberg spin chain.

1.1.3   Ferromagnetism

First attempt of theoretical description of ferromagnetic materials can be attributed to Weiss [84]. He assumed, that the field generated by the material itself is enough to fully saturate it magnetically and thus ferromagnets are always saturated (before one accounts for thermal vibrations). To explain the possibility of varied degrees of magnetization (including a lack of external field) of pieces of iron and other ferromagnetic materials, he proposed that their interior is divided into so called domains

(each fully saturated), which can cancel each other out. He even correctly predicted the two possible magnetization processes: through the change of domain size and change in direction of their magnetization. Weiss's theory was good enough to explain the Curie point and together with Pierre Curie he was able to properly describe behavior of ferromagnets above the critical temperature, yet he could not find a reason for the internal magnetization he proposed. This feat was achieved by Heisenberg [85] in 1928 (see also his later works [86, 87]). Under the Heitler-London approximation he has shown, that the Pauli exclusion principle can lead to an ordered ground state for valence electrons in crystals. Moreover, he predicted, that ferromagnetism in elementally pure materials (the only case he was able to properly solve) requires specific crystal configuration, in which each atom has at least 8 nearest neighbours. This was a significant breakthrough, since not only an underlying phenomenon responsible for ferromagnetism was found, but also the theory properly shown, which elements have ferromagnetic properties. Heisenberg's ideas were further refined by Bloch [88, 89], who was able to introduce thermodynamics into the model and thus calculate the magnetic properties of ferromagnets below and above the Curie point. More importantly, he was the first to obtain a model of a domain wall from first principles. His estimations of domain size and shape based on microscopic qualities of a given material were unfortunately incomplete due to the importance of magnetostatic energy and the macroscopic geometry of a particular ferromagnetic object as was pointed by Landau and Lifschitz in their seminal paper [90] in 1935. Their most important achievement was construction of a classical system describing key qualitative and quantitative properties of ferromagntic materials. It uses a classical expression for internal energy density (here with an easy axis anisotropy with z as the preferred direction of spin as per the original paper)

$$H = \int \left\{ \frac{1}{2\alpha} \left[(\nabla s_x)^2 + (\nabla s_y)^2 + (\nabla s_z)^2 \right] + \frac{1}{2\beta} \left[s_x^2 + s_y^2 \right] \right\} dV, \qquad (1.1.44)$$

where $\overrightarrow{s} = [s_x, s_y, s_z]$ is a vector representing local magnetization (in a discrete version of the model it would represent magnetic moment of a single atom), ∇ the standard vector differential operator, α parameter can be derived from Heisenberg's exchange integrals or experimentally from Curie point of a given material and β is the anisotropy parameter. As such this model allows one to calculate the spatial parameters of the domain wall as well as its energy. Yet, to properly calculate the shape and distribution of domain walls, Landau and Lifschitz noted, that the global magnetostatic energy has to be taken into account. By imposing a lack of external field generated by a piece of ferromagnet of a given shape, they were able to replicate experimental data on that subject. The approach presented by Landau and Lifschitz was adapted and refined by numerous researchers in following decades. A very comprehensive summary of early accomplishments in the field of magnetic domain walls was given by Kittel in 1949 [91]. In later years the dynamic properties of magnetic systems were more closely studied and in 1966 Akhiezer and Borovik have presented a throughout study of spin waves in ferromagnets [92]. One of the most interest-

ing finds in their work is that solitary waves have a specific maximum propagation speed. Other researchers enriched the system by accounting for external magnetic field (see for example [14, 15]), non-zero temperature [93], impurities in the crystal structure [94], long-range interactions [95], discrete nature of the real ferromagnets in form of perturbative corrections [96] or numerical study of an explicitly discrete system [97]. A throughout overview of the fully quantum Heisenberg chain was given by Maillet [98] with focus on connection between theory and experimental data.

1.2   Aim of work

The main goal of this work is to obtain an analytic form of non-relativistic quasiclassical corrections to energy of static, periodic solution of ϕ^4 nonlinear model as well as the static quasi-periodic ($\varphi(x + x_p) = \varphi(x) + C$) solution of Sine-Gordon model with inclusion of all due rest variables. Until now, there was very little progress in quantising nonlinear fields other than single static solitons. In order to achieve this objective refinement of mathematical tools used in semiclassical quantization is in order. First point of order is a revision and rederivation of zeta-function regularization scheme. It is required to properly determine the meaning of the so called mass scale and the way of estimating its value. It is equally important to extract as much of physical data as possible from the general energy corrections formula without explicitly solving the inherent eigenvalue problem. Another goal is to refine the mathematical methods used to construct the generalized zeta-function. Most notably this thesis contains a detailed analysis of the Drach equation including the necessary conditions on potentials to have finite gap spectrum (through a connection between borders of energy bands and singularities of the Green function) as well as a proof of uniqueness of solutions for such potentials. The final goal is to apply the mathematical results to the domain walls of the classical Heisenberg magnetic chain in easy axis and easy plane approximation. Considering the relatively high ratio of quantum corrections to classical energy, measurable effects on domain wall dynamics are expected. In particular the dependence of energy on the spatial period of analysed field contains valuable information on interaction energy of domain walls. Negative sign of energy corrections in a given set of system parameters can also lead to spontaneous domain wall creation, which could not be accounted for in the classical system.

1.3   Thesis Outline

Second chapter is devoted to the classical Heisenberg magnetic chain model with external magnetic field perpendicular to the chain. It starts with an explanation of relation between the discrete and continuous chain, from which two distinct approximations are derived for ferromagnets with axial anisotropy: one for the so called easy

axis case, where spin direction in line with the anisotropy is preferred, and one for the so called easy plane case, where spin direction perpendicular to the anisotropy is preferred.

Third chapter provides an in-depth overview of the semiclassical quantization in version developed by Maslov. Firstly the original definition of path integrals is discussed and the difference between the quantum-mechanical case and field theory path integrals is shown. Next the connection between the original definition and formula given by Maslov is explained and the general form of corrections to energy is provided. Lastly the zeta-function regularization scheme is discussed and the way in which variable separation simplifies multidimensional problems is shown.

Fourth chapter focuses on the Drach equation in the context of Green function problem. It opens with a proof, that diagonal elements of a given Green function problem solve the Drach equation. Following is a detailed analysis of the equation itself and resulting set of conditions for finite gap potentials. Furthermore, the chapter contains a proof of uniqueness of solutions representing Green function diagonal for finite gap potentials and a solving algorithm (sample implementation in appendix B). It concludes with a brief discussion of scaling operation and their impact on solutions.

The last chapter concentrates on calculating energy corrections to static, periodic solution of ϕ^4 nonlinear model as well as the static quasi-periodic solution of Sine-Gordon model, which contain the single kink solution as a limit. It begins with an analysis of scaling operations on the general formulae for semiclassical corrections, which allows for extraction of physically relevant data without the need of solving the underlying spectral problem.

Chapter 2

Heisenberg's magnetic chain

2.1 General equations of motion

Full set of quantum-mechanical equations for a large number of electrons is immensely difficult to handle without approximations. Heisenberg managed to derive the phenomenon of ferromagnetism from first principles [85], but in many cases one has to either resort to time consuming numerical procedures or vastly reduced systems. Therefore in the study of various aspects of magnetic structure of solids classical systems are created from their quantum counterparts. In case of magnetic domain walls, one often is concerned with a single chain of atoms with a single relevant electron on each of them and models a cross-section of a domain wall as a soliton (or a similar solution) on this chain [14, 15]. Spins are assumed to be simple localized vectors of unitary length. Assuming an axial anisotropy (here in line with the modelled chain, but it is not essential) and an external magnetic field perpendicular to the anisotropy direction we will work with a Hamiltonian of form

$$H = -J\sum_n \vec{s}_n \vec{s}_{n+1} + D\sum_n (s_{n,3})^2 - g\mu_B B \sum_n s_{n,1} \tag{2.1.1}$$

with $\vec{s}_n = [s_{n,1}, s_{n,2}, s_{n,3}]$ (since we impose an additional limit of $|\vec{s}_n| = 1$ only two of those elements will be independent) representing electron spins, D as anisotropy parameter, g as the g-factor, μ_B as Bohr magneton, B as the amplitude of external magnetic field and J representing the so-called exchange energy arising from the influence of Pauli exclusion principle on interaction of electrons, which can be calculated from first principles [85] or estimated from Weiss molecular field model [99]. Equation parameters can also be obtained experimentally as was shown in [90]. According to [100] the equation of motion for this model can be written as

$$\hbar\partial_t \vec{s}_n = \vec{s}_n \times \nabla_{s_{n,j}} H, \tag{2.1.2}$$

where $\vec{\nabla}_{s_n} = \left[\partial_{s_{n,1}}, \partial_{s_{n,2}}, \partial_{s_{n,3}}\right]$. We will now insert the explicit form of

$$\vec{\nabla}_{s_n} H = -J(\vec{s}_{n+1} + \vec{s}_{n-1}) + 2Ds_{n,3}\hat{x}_3 - g\mu_B B\hat{x}_1 \tag{2.1.3}$$

into (2.1.2)

$$\hbar\partial_t \overrightarrow{s}_n = \overrightarrow{s}_n \times (-J(\overrightarrow{s}_{n+1} + \overrightarrow{s}_{n-1}) + 2Ds_{n,3}\hat{x}_3 - g\mu_B B\hat{x}_1). \tag{2.1.4}$$

We will procede with writing the equations explicitly for each base direction

$$\begin{cases} \hbar\partial_t s_{n,1} = -J[s_{n,2}(s_{n+1,3} + s_{n-1,3}) - s_{n,3}(s_{n+1,2} + s_{n-1,2})] + 2Ds_{n,2}s_{n,3} \\ \hbar\partial_t s_{n,2} = -J[s_{n,3}(s_{n+1,1} + s_{n-1,1}) - s_{n,1}(s_{n+1,3} + s_{n-1,3})] - 2Ds_{n,1}s_{n,3} - g\mu_B Bs_{n,3} \\ \hbar\partial_t s_{n,3} = -J[s_{n,1}(s_{n+1,2} + s_{n-1,2}) - s_{n,2}(s_{n+1,1} + s_{n-1,1})] + g\mu_B Bs_{n,2} \end{cases}$$
$$\tag{2.1.5}$$

and taking the continuum limit (with a as lattice constant and $\overrightarrow{s} = [s_1, s_2, s_3]$ as a smooth function in t and x_3)

$$\begin{cases} \hbar\partial_t s_1 = -J[s_2(2s_3 + a^2\partial_{x_3}^2 s_3) - s_3(2s_2 + a^2\partial_{x_3}^2 s_2)] + 2Ds_2 s_3 \\ \hbar\partial_t s_2 = -J[s_3(2s_1 + a^2\partial_{x_3}^2 s_1) - s_1(2s_3 + a^2\partial_{x_3}^2 s_3)] - 2Ds_1 s_3 - g\mu_B Bs_3 \\ \hbar\partial_t s_3 = -J[s_1(2s_2 + a^2\partial_{x_3}^2 s_2) - s_2(2s_1 + a^2\partial_{x_3}^2 s_1)] + g\mu_B Bs_2. \end{cases} \tag{2.1.6}$$

Considering the fact that $|\overrightarrow{s}| = 1$ we can work with two unknown functions (angles in three-dimensional space) instead of three (s_1, s_2 and s_3). With this in mind, we substitute

$$\overrightarrow{s} = [\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi] \tag{2.1.7}$$

and after simple reduction we obtain general equations of motion

$$\begin{cases} \hbar\cos\phi\,\partial_t\theta = Ja^2[\partial_{x_3}^2\phi + \sin\phi\cos\phi(\partial_{x_3}\theta)^2] - 2D\cos\phi\sin\phi - g\mu_B B\sin\phi\cos\theta \\ \hbar\partial_t\phi = -Ja^2[\cos\phi\,\partial_{x_3}^2\theta - 2\sin\phi\,\partial_{x_3}\theta\,\partial_{x_3}\phi] + g\mu_B B\sin\theta \end{cases}$$
$$\tag{2.1.8}$$

as well as the Hamiltonian

$$\begin{aligned} H = & -\frac{J}{a}\{\cos\phi\cos(a\partial_{x_3}\theta)[\cos\phi\cos(a\partial_{x_3}\phi) - \sin\phi\sin(a\partial_{x_3}\phi)] \\ & + \sin\phi[\sin\phi\cos(a\partial_{x_3}\phi) + \sin(a\partial_{x_3}\phi)\cos\phi]\} \\ & + \frac{2D}{a}\sin\phi - \frac{g\mu_B B}{a}\cos\theta\cos\phi \end{aligned} \tag{2.1.9}$$

which might require addition of a constant to set the energy minimum at 0. In the next two subsections we will reproduce two distinctive simplifications of this system.

2.2 Easy plane approximation

If the axial anisotropy parameter D is greater than zero, then spins lying in the OX_1X_2 plane are preferred. Without the external magnetic field, spins would be able to rotate freely in that plane, thus the name of the approximation, which takes it a step further and assumes, that ϕ angle is almost zero. For this approximation to be viable, axial anisotropy has to be the dominating factor in determining spin direction. Since the nearest neighbour interaction term of the Hamiltonian (the part

with coefficient J) doesn't prefer any direction, it will suffice to assume $D >> g\mu_B B$. From the assumption $\phi \approx 0$ we can readily obtain $\sin(\phi) \approx \phi$ and $\cos(\phi) \approx 1$. In essence, we treat ϕ as the least significant value. After inserting those approximation to equations of motion we obtain

$$\begin{cases} \hbar\partial_t\theta &= Ja^2[\partial_{x_3}^2\phi + \phi(\partial_{x_3}\theta)^2] - 2D\phi - g\mu_B B\phi\cos\theta \\ \hbar\partial_t\phi &= -Ja^2[\partial_{x_3}^2\theta - 2\phi\partial_{x_3}\theta\partial_{x_3}\phi] + g\mu_B B\sin\theta. \end{cases} \quad (2.2.10)$$

Considering the continuum approximation we can omit all components containing $a^2\partial_{x_3}^2\phi$ and $a^2\phi$ as a product of two least significant values. We arrive at

$$\begin{cases} \hbar\partial_t\theta &= -2D\phi - g\mu_B B\phi\cos\theta \\ \hbar\partial_t\phi &= -Ja^2\partial_{x_3}^2\theta + g\mu_B B\sin\theta. \end{cases} \quad (2.2.11)$$

Now we use our base assumption $D >> g\mu_B B$ on the first equation to obtain

$$\begin{cases} \hbar\partial_t\theta &= -2D\phi \\ \hbar\partial_t\phi &= -Ja^2\partial_{x_3}^2\theta + g\mu_B B\sin\theta. \end{cases} \quad (2.2.12)$$

As can be seen, we can use the first equation to eliminate ϕ from the second one and obtain an equation of motion for θ angle only

$$\begin{cases} \phi = -\frac{\hbar}{2D}\partial_t\theta \\ \frac{\hbar^2}{2D}\partial_t^2\theta = Ja^2\partial_{x_3}^2\theta - g\mu_B B\sin\theta \end{cases} \quad (2.2.13)$$

with a corresponding Hamiltonian

$$H = \frac{\hbar^2}{4aD}\left(\frac{\partial\theta}{\partial t}\right)^2 + \frac{Ja}{2}\left(\frac{\partial\theta}{\partial x_3}\right)^2 + \frac{g\mu_B B}{a}(1 - \cos\theta) \quad (2.2.14)$$

with a constant $\frac{g\mu_B B}{a}$ added to set the energy minimum at 0. This system can be easily identified as the Sine-Gordon equation. We will now proceed to derive a quasi-periodic solution of this equation describing the so-called stripe domains (assuming the same width of each domain as well as the same direction each domain wall is twisted in) and an asymptotic case thereof describing a singular ferromagnetic domain wall. Let us start with the equation of motion (2.2.13) in a dimensionless form ($t = Tt'$ and $x_3 = ax'$ with T as a parameter introduced in the action integral in Chapter 3)

$$\frac{\hbar^2}{2JDT^2}\partial_{t'}^2\theta = \partial_{x'}^2\theta - \frac{g\mu_B B}{J}\sin\theta. \quad (2.2.15)$$

In this form, there are only two relevant parameters: phase speed

$$c = \sqrt{\frac{2JDT^2}{\hbar^2}} \quad (2.2.16)$$

and potential amplitude

$$m_\theta^2 = \frac{g\mu_B B}{J}. \quad (2.2.17)$$

We will use this notation for simplicity

$$\frac{1}{c^2}\partial_{t'}^2\theta = \partial_{x'}^2\theta - m_\theta^2\sin\theta. \tag{2.2.18}$$

As we are only concerned with static solutions (due to limitations of Green function evaluation methods and a few concerns about the quantization scheme), we can simplify the equation to

$$\partial_{x'}^2\theta - m_\theta^2\sin\theta = 0. \tag{2.2.19}$$

From here we multiply the equation by $\partial_{x'}\theta$

$$\partial_{x'}^2\theta\partial_{x'}\theta - m_\theta^2\sin\theta\partial_{x'}\theta = 0 \tag{2.2.20}$$

and integrate over x'

$$\frac{1}{2}\left(\partial_{x'}\theta\right)^2 + m_\theta^2\cos\theta = C, \tag{2.2.21}$$

where C is an arbitrary integration constant. We can now substitute

$$\theta = 2\arcsin\theta' + \pi, \tag{2.2.22}$$

which combined with the Pythagorean identity gives us

$$2\frac{(\partial_{x'}\theta')^2}{1-\theta'^2} = C + m_\theta^2 - 2m_\theta^2\theta'^2. \tag{2.2.23}$$

After multiplication by $\frac{1}{2}(1-\theta'^2)$ and a substitution

$$k^2 = \frac{C + m_\theta^2}{2m_\theta^2} \tag{2.2.24}$$

we arrive at

$$(\partial_{x'}\theta')^2 = m_\theta^2(1-\theta'^2)(k^2-\theta'^2), \tag{2.2.25}$$

which is the equation defining the Jacobi sn function. Namely

$$\theta'(x') = k\,\text{sn}(m_\theta x'; k). \tag{2.2.26}$$

With this we can build the actual solution of Sine-Gordon equation

$$\theta(x') = 2\arcsin[k\,\text{sn}(m_\theta x'; k)] + \pi \tag{2.2.27}$$

or

$$\theta(x_3) = 2\arcsin\left[k\,\text{sn}\left(\sqrt{\frac{g\mu_B B}{J}}\frac{x_3}{a}; k\right)\right] + \pi, \tag{2.2.28}$$

if we use the original notation. It is important to note, that this solution without additional care is only valid for a half-period of the sn function due to the nature of the arc sin function, which should be in general treated as a multivalued function. Yet if we start from any arbitrary part of the solution there is always a unique smooth continuation. In the end we obtain a so called quasi-periodic solutions, which is

characterized by $\theta\left(x_3 + \frac{2}{m_\theta}\mathcal{K}(k)\right) = \theta(x_3) + 2\pi$ quality. In the $k \to 1$ limit this solution will converge to a well known single soliton:

$$\theta(x_3) = 2\arcsin\left[\mathrm{tgh}\left(\sqrt{\frac{g\mu_B B}{J}}\frac{x_3}{a}\right)\right] + \pi. \tag{2.2.29}$$

Energy per single period of the solution ($\frac{2}{m_\theta}\mathcal{K}(k)$), which represents a single domain wall, can be calculated by integration of the Hamiltonian

$$E_c(k) = \frac{4}{k^2}\sqrt{Jg\mu_B B}\left[\left(1 + k^2\right)\mathcal{E}(k) + \left(1 - k^2\right)\mathcal{K}(k)\right], \tag{2.2.30}$$

where $\mathcal{K}$ denotes the complete elliptic integral of the first kind and $\mathcal{E}$ the complete elliptic integral of the second kind. In the case of a single domain wall ($k \to 1$) it simplifies to

$$E_c = 8\sqrt{Jg\mu_B B}. \tag{2.2.31}$$

2.3 Easy axis approximation

If the anisotropy coefficient D is negative, then without the external magnetic field spin direction in line with the anisotropy axis is preferred. In extreme cases, when nearest neighbour interaction is weak (which would invalidate the continuous approximation) it would result in a discrete Ising model. Here we will focus on a different regime. Let us look on the stationary points ($(\theta, \phi) = const$) of an easy axis magnetic chain ($D < 0$) with external magnetic field perpendicular to the anisotropy axis.

$$\begin{cases} 0 = -2D\cos\phi\sin\phi - g\mu_B B\sin\phi\cos\theta \\ 0 = g\mu_B B\sin\theta, \end{cases} \tag{2.3.32}$$

which gives us readily

$$\begin{cases} \theta &=& 0 \\ \phi &\in& \{-\arccos(\frac{g\mu_B B}{-2D}), 0, \arccos(\frac{g\mu_B B}{-2D})\} \end{cases} \tag{2.3.33}$$

with $\phi = 0$ being unstable. For $g\mu_B B$ close to but lower than $-2D$ both stable points are close to $\phi = 0$. In such a situation it is valid to assume $\phi \approx 0$ with ϕ^3 as the highest considered term due to the instability of the $\phi = 0$ point and with θ as the highest considered term

$$\begin{cases} \hbar\partial_t\theta &=& Ja^2\partial_{x_3}^2\phi - 2D(\phi - \frac{2\phi^3}{3}) - g\mu_B B(\phi - \frac{\phi^3}{6}) \\ \hbar\partial_t\phi &=& -Ja^2\partial_{x_3}^2\theta + g\mu_B B\theta. \end{cases} \tag{2.3.34}$$

As in the easy plane case we consider $a^2\partial_{x_3}^2\theta$ as a product of two least significant values and omit it

$$\begin{cases} \hbar\partial_t\theta &=& Ja^2\partial_{x_3}^2\phi - 2D(\phi - \frac{2\phi^3}{3}) - g\mu_B B(\phi - \frac{\phi^3}{6}) \\ \hbar\partial_t\phi &=& g\mu_B B\theta, \end{cases} \tag{2.3.35}$$

which leads to

$$
\begin{cases}
\frac{\hbar^2}{g\mu_B B}\partial_t^2\phi = Ja^2\partial_{x_3}^2\phi - (2D + g\mu_B B)\phi + \frac{8D+g\mu_B B}{6}\phi^3 \\
\theta = \frac{\hbar}{g\mu_B B}\partial_t\phi.
\end{cases}
\tag{2.3.36}
$$

The result represents the ϕ^4 model with the energy density

$$
\begin{aligned}
H \;=\;& \frac{\hbar^2}{2ag\mu_b B}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{Ja}{2}\left(\frac{\partial\phi}{\partial x_3}\right)^2 \\
& - \frac{3(2D + g\mu_B B)^2}{2a(8D + g\mu_B B)} + \frac{2D + g\mu_B B}{2a}\phi^2 - \frac{8D + g\mu_B B}{24a}\phi^4
\end{aligned}
\tag{2.3.37}
$$

with $-\frac{3(2D+g\mu_B B)^2}{2a(8D+g\mu_B B)}$ added to set the energy minimum at 0. Now we will derive a periodic solution of this system describing the so called stripe domains, which similarly to the easy plane case will have a single domain wall solution as a limit. As before, we will start by introducing dimensionless variables $t = Tt'$ and $x_3 = ax'$ (with a and T having the same meaning as before)

$$
\frac{\hbar^2}{g\mu_B B J T^2}\partial_{t'}^2\phi = \partial_{x'}^2\phi - \frac{2D + g\mu_B B}{J}\phi + \frac{8D + g\mu_B B}{6J}\phi^3.
\tag{2.3.38}
$$

We can readily identify the phase speed

$$
c = \frac{T}{\hbar}\sqrt{g\mu_B B J}.
\tag{2.3.39}
$$

Since we are interested only in a static solution, we can omit the time derivative

$$
\partial_{x'}^2\phi - \frac{2D + g\mu_B B}{J}\phi + \frac{8D + g\mu_B B}{6J}\phi^3 = 0
\tag{2.3.40}
$$

and as in the previous case multiply the equation by $\partial_{x'}\phi$, and integrate it over x'

$$
\frac{1}{2}(\partial_{x'}\phi)^2 = \frac{2D + g\mu_B B}{2J}\phi^2 - \frac{8D + g\mu_B B}{24J}\phi^4 + C.
\tag{2.3.41}
$$

For brevity, we will introduce

$$
\begin{cases}
m_\phi &=\; \sqrt{-\frac{2D+g\mu_B B}{J}} \\
V &=\; \sqrt{\frac{12(2D+g\mu_B B)}{8D+g\mu_B B}},
\end{cases}
\tag{2.3.42}
$$

which will result in

$$
(\partial_{x'}\phi)^2 = -m_\phi^2\phi^2 + \frac{m_\phi^2}{V^2}\phi^4 + 2C.
\tag{2.3.43}
$$

From here we can again refer to the definition of Jacobi sn function and substitute a solution of form $\phi = C_1\mathrm{sn}(C_2 x'; k)$ with C_1 and C_2 as constant to be found after inserting the solution back to (2.3.43). We will obtain

$$
\phi(x') = \sqrt{\frac{k^2}{1 + k^2}}V\mathrm{sn}\left(\frac{m_\phi}{\sqrt{1 + k^2}}x'; k\right)
\tag{2.3.44}
$$

with k as a free parameter. For $k \to 1$ it reduces to a well known single kink solution

$$\phi(x') = \frac{1}{\sqrt{2}} V \tgh \left(\frac{m_\phi}{\sqrt{2}} x' \right). \tag{2.3.45}$$

With this we can also calculate the classical energy per half period of the solution $2\frac{\sqrt{1+k^2}}{m_\phi}\mathcal{K}(k)$, which represents a single domain wall

$$E_c = \frac{J m_\phi V^2}{12(1 + k^2)^{\frac{3}{2}}} \left[8(1 + k^2)\mathcal{E}(k) + (-5 + 2k^2 + 3k^4)\mathcal{K}(k) \right], \tag{2.3.46}$$

which for $k = 1$ simplifies to

$$E_c = \frac{\sqrt{2}}{3} J m_\phi V^2, \tag{2.3.47}$$

or

$$E_c = \frac{4\sqrt{-2J}(2D + g\mu_B B)^{\frac{3}{2}}}{8D + g\mu_B B} \tag{2.3.48}$$

in the original notation.

Chapter 3

Semiclassical quantization

3.1 Feynman formalism

For the purpose of this section we will use Dirac notation for quantum states. More specifically $|x_n\rangle$ will denote a state fully localized in point x_n. In Feynman formulation of quantum mechanics [45] propagation operator can be expressed as [101] (conditional probability densities were obviously expressed in many earlier publications including the original [45, 61], but we are using this form of the left hand side for elegance)

$$\langle x_T| \, e^{-\frac{i}{\hbar}TH} \, |x_0\rangle = \lim_{N\to\infty} \int_{-\infty}^{\infty} \prod_{n=1}^{N} e^{\frac{i}{\hbar}S(x_{n-1},x_n,\frac{T}{N})} \prod_{n=1}^{N-1} \frac{dx_N}{\chi} \qquad (3.1.1)$$

assuming $x_N = x_T$ and that $x_n \in \mathbb{R}$, where $S(x_{n-1}, x_n, \frac{T}{N})$ represents an action integral over a classical path between points x_{n-1} and x_n with time difference $\frac{T}{N}$ (assuming a conservative system) and χ is a normalising factor as per the original definition [45]. Assuming the most typical form of action integral for a material point of mass M

$$S(x,T) = \int_0^T \left[\frac{M}{2} \left(\frac{\partial x}{\partial t} \right)^2 - V(x) \right] dt \qquad (3.1.2)$$

(with x as a function of time with specified boundary conditions) we can simplify (3.1.1) for $N \to \infty$, since for short time intervals classical trajectory can be approximated with a straight line between (x_{n-1}, t_{n-1}) and (x_n, t_n) provided the potential is smooth

$$S\left(x_{n-1}, x_n, \frac{T}{N}\right) \approx \frac{MN}{2T}(x_n - x_{n-1})^2 - \frac{T}{N}V(x_n). \qquad (3.1.3)$$

This approximation should still hold for almost everywhere smooth or discontinuous potentials (as long as the set of discontinuities has no accumulation points), but they should be treated with additional care. The gaussian term becomes dominant as $N \to \infty$. Considering that $\Delta x_n = x_n - x_{n-1}$ are independent variables, this formulation of Feynman integral is similar to that of a Wiener process, yet there

are substantial differences as was shown in [102]. At this point one can properly define the normalising factor using the fact, that the integral has to converge in the $N \to \infty$ limit. Namely for high enough N we should have

$$e^{\frac{iMN}{2T\hbar}(x_n - x_{n-1})^2} = \int_{-\infty}^{\infty} e^{\frac{iMN}{T\hbar}[(x - x_{n-1})^2 + (x_n - x)^2]} \frac{dx}{\chi}. \tag{3.1.4}$$

This leads directly to

$$\chi = \sqrt{\frac{i\pi T\hbar}{MN}} \tag{3.1.5}$$

for a time interval $\frac{T}{N}$. It is of note, that the convergence requirement automatically implies that the propagator always converges to identity operator for small T. The same result was obtained by Feynman in [45] by matching path integral formalism with Schrödinger viewpoint. Difference by a factor of $\sqrt{2}$ stems from the fact, that Feynman integrated over Δx_n instead of x_n. This result can be easily generalized to d-dimensional theory, where we would take the same normalising factor for each degree of freedom (as long as they are independent).

The same ideas could potentially be used to quantize field theories

$$\langle \psi_T | e^{-\frac{i}{\hbar}TH} | \psi_0 \rangle = \lim_{N \to} \int_{C(\mathbb{R})} \prod_{n=1}^{N} e^{\frac{i}{\hbar}S(\psi_{n-1}, \psi_n, \frac{T}{N})} \prod_{n=1}^{N-1} \frac{D\psi_n}{\chi}, \tag{3.1.6}$$

where ψ are twice differentiable fields over $\mathbb{R}^2$ (position variable x and time variable t), ψ_n denote positions at given time points (as in $\psi(t_n, x) \equiv \psi_n(x)$), $\psi_N = \psi_T$ and $S(\psi_{n-1}, \psi_n, \frac{T}{N})$ represents an action over a classical path between ψ_{n-1} and ψ_n in time $\frac{T}{N}$ and χ is a normalising factor. Assuming action integral

$$S(\psi, T) = \int_0^T \int_{\mathbb{R}} \left[\frac{M}{2} \left(\frac{\partial \psi}{\partial t} \right)^2 - \frac{G}{2} \left(\frac{\partial \psi}{\partial x} \right)^2 - V(\psi) \right] dx \, dt \tag{3.1.7}$$

we can obtain a similar convergence condition as for a quantum mechanical path integral. Namely for $N \to \infty$

$$S\left(\psi_{n-1}, \psi_n, \frac{T}{N} \right) \approx \int_{\mathbb{R}} \left\{ \frac{MN}{2T} [\psi_n - \psi_{n-1}]^2 \right.$$
$$\left. - \frac{G}{2} \int_0^{\frac{T}{N}} \left[\frac{tN}{T} \frac{\partial \psi_n}{\partial x} - \left(1 - \frac{tN}{T} \right) \frac{\partial \psi_{n-1}}{\partial x} \right]^2 dt \right\} dx \tag{3.1.8}$$

and subsequently

$$e^{\frac{i}{\hbar}S\left(\psi_{n-1}, \psi_n, \frac{T}{N}\right)} \approx \int_{C(\mathbb{R})} e^{\frac{i}{\hbar} \int_{\mathbb{R}} \left\{ \frac{MN}{T} [\psi_{n-1/2} - \psi_{n-1}]^2 \right\} dx}$$
$$\times e^{\frac{i}{\hbar} \int_{\mathbb{R}} \left\{ -\frac{G}{2} \int_0^{\frac{T}{2N}} \left[\frac{2tN}{T} \frac{\partial \psi_{n-1/2}}{\partial x} - \left(1 - \frac{2tN}{T} \right) \frac{\partial \psi_{n-1}}{\partial x} \right]^2 dt \right\} dx}$$
$$\times e^{\frac{i}{\hbar} \int_{\mathbb{R}} \left\{ \frac{MN}{T} [\psi_n - \psi_{n-1/2}]^2 \right\} dx}$$
$$\times e^{\frac{i}{\hbar} \int_{\mathbb{R}} \left\{ -\frac{G}{2} \int_0^{\frac{T}{2N}} \left[\frac{2tN}{T} \frac{\partial \psi_n}{\partial x} - \left(1 - \frac{2tN}{T} \right) \frac{\partial \psi_{n-1/2}}{\partial x} \right]^2 dt \right\} dx} \frac{D\psi_{n-1/2}}{\chi}. \tag{3.1.9}$$

After integration over t we obtain

$$
\begin{aligned}
e^{\frac{i}{\hbar}S\left(\psi_{n-1},\psi_n,\frac{T}{N}\right)} \approx \int_{C(\mathbb{R})} & e^{\frac{i}{\hbar}\int_{\mathbb{R}}\left\{\frac{MN}{T}[\psi_{n-1/2}-\psi_{n-1}]^2\right\}dx} \\
\times & e^{\frac{i}{\hbar}\int_{\mathbb{R}}\left\{-\frac{G}{2}\int_0^{\frac{T}{2N}}\left[\left(\frac{\partial\psi_{n-1/2}}{\partial x}\right)^2-\frac{\partial\psi_{n-1/2}}{\partial x}\frac{\partial\psi_{n-1}}{\partial x}+\left(\frac{\partial\psi_{n-1}}{\partial x}\right)^2\right]\right\}dx} \\
\times & e^{\frac{i}{\hbar}\int_{\mathbb{R}}\left\{\frac{MN}{T}[\psi_n-\psi_{n-1/2}]^2\right\}dx} \\
\times & e^{\frac{i}{\hbar}\int_{\mathbb{R}}\left\{-\frac{G}{2}\int_0^{\frac{T}{2N}}\left[\left(\frac{\partial\psi_{n-1/2}}{\partial x}\right)^2-\frac{\partial\psi_{n-1/2}}{\partial x}\frac{\partial\psi_n}{\partial x}+\left(\frac{\partial\psi_n}{\partial x}\right)^2\right]\right\}dx}\frac{D\psi_{n-1/2}}{\chi}.
\end{aligned}
$$

$$(3.1.10)$$

In this case, one does not have an easy choice for the normalising factor χ. In fact, normalization of such functional gaussian integrals (real or complex) is still an open question.

3.2   Maslov representation

A different approach was presented by Maslov in [60, 61]. Let us expand the action integral into a Taylor series around a specific classical field φ over $\mathbb{R}^2 \times [0, al]^{d-1}$ (one time variable t spanning over $\mathbb{R}$ and d spatial variables written summarily as $\overrightarrow{x} = [x_1, ..., x_d]$ with x_1 spanning over $\mathbb{R}$ and all others over $[0, al]$) with a given boundary conditions (Dirichlet or Neumann) or periodic condition. Namely, we use a substitution

$$\phi = \varphi + \sum_j w_j \phi_j \qquad (3.2.11)$$

with ϕ_j fulfilling the same type of condition as φ with the difference, that in case of boundary conditions its value (or derivative) will vanish at the boundary. If we use the chosen base for the Taylor expansion

$$S(\phi) = S(\varphi) + \frac{1}{2}\sum_{j,k}\frac{\partial^2 S}{\partial w_j \partial w_k}(\varphi)w_j w_k + \ldots, \qquad (3.2.12)$$

we obtain a following approximation of propagation operator

$$\int_{C^{0,T}_{\psi_0,\psi_T}} e^{\frac{i}{\hbar}S(\phi)}D\phi \approx \int_{\mathbb{R}} e^{\frac{i}{\hbar}\left[S(\varphi)+\frac{1}{2}\sum_{j,k}\frac{\partial^2 S}{\partial w_j \partial w_k}(\varphi)w_j w_k\right]}\prod_f dw_f \qquad (3.2.13)$$

with φ as the classical path with boundary conditions $\varphi(0, \overrightarrow{x}) \equiv \psi_0(\overrightarrow{x})$, $\varphi(T, \overrightarrow{x}) \equiv \psi_T(\overrightarrow{x})$ and ϕ_j as a base of all functions over $\mathbb{R}^2 \times [0, al]^{d-1}$ fulfilling conditions $\forall_j \phi_j(0, \overrightarrow{x}) \equiv \phi_j(T, \overrightarrow{x}) \equiv 0$. It is easy to see, that the classical action can be extracted out of the integral

$$e^{\frac{i}{\hbar}S(\varphi)}\int_{\mathbb{R}} e^{\frac{i}{2\hbar}\sum_{j,k}\frac{\partial^2 S}{\partial w_j \partial w_k}(\varphi)w_j w_k}\prod_f dw_f. \qquad (3.2.14)$$

Assuming (specifics of the spatial domain are not important at this point)

$$S(\phi) = \int_0^T \int_{\mathbb{R} \times [0,al]^{d-1}} \left[\frac{\rho}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 - \sum_{n=1}^d \frac{G_n}{2} \left(\frac{\partial \phi}{\partial x_n} \right)^2 - V(\phi) \right] d\overrightarrow{x}\, dt \qquad (3.2.15)$$

we obtain

$$\frac{\partial S}{\partial w_j \partial w_k}(\varphi) = \int_0^T \int_{\mathbb{R} \times [0,al]^{d-1}} \left[\rho \frac{\partial \phi_j}{\partial t} \frac{\partial \phi_k}{\partial t} \right.$$
$$\left. - \sum_{n=1}^d G_n \frac{\partial \phi_j}{\partial x_n} \frac{\partial \phi_k}{\partial x_n} - V''(\varphi)\phi_j\phi_k \right] d\overrightarrow{x}\, dt. \qquad (3.2.16)$$

Considering the boundary conditions

$$\frac{\partial S}{\partial w_j \partial w_k}(\varphi) = \int_0^T \int_{\mathbb{R} \times [0,al]^{d-1}} \left[-\rho \frac{\partial^2 \phi_j}{\partial t^2} \phi_k \right.$$
$$\left. + \sum_{n=1}^d G_n \frac{\partial^2 \phi_j}{\partial x_n^2} \phi_k - V''(\varphi)\phi_j\phi_k \right] d\overrightarrow{x}\, dt. \qquad (3.2.17)$$

It is important to note, that Robin boundary conditions would generate additional terms due to integration by parts. By introducing dimensionless variables $\overrightarrow{x} = a\overrightarrow{x'}$, $t = Tt'$,

$$\frac{\partial S}{\partial w_j \partial w_k}(\varphi) = Ta^d \int_0^1 \int_{\mathbb{R} \times [0,l]^{d-1}} \left[-\frac{\rho}{T^2} \frac{\partial^2 \phi_j}{\partial t'^2} \phi_k \right.$$
$$\left. + \sum_{n=1}^d \frac{G_n}{a^2} \frac{\partial^2 \phi_j}{\partial x_n'^2} \phi_k - V''(\varphi)\phi_j\phi_k \right] d\overrightarrow{x'}\, dt' \qquad (3.2.18)$$

and inserting the result back into (3.2.14)

$$e^{\frac{i}{\hbar}S(\varphi)} \int_{\mathbb{R}} e^{\frac{iTa^d}{2\hbar} \sum_{j,k} \int_0^1 \int_{\mathbb{R} \times [0,l]^{d-1}} \left(-\frac{\rho}{T^2} \frac{\partial^2 \phi_j}{\partial t^2} \phi_k + \sum_{n=1}^d \frac{G_n}{a^2} \frac{\partial^2 \phi_j}{\partial x_n'^2} \phi_k - V''(\varphi)\phi_j\phi_k \right) d\overrightarrow{x'} dt' w_j w_k} \prod_f dw_f$$
$$(3.2.19)$$

we can rewrite the integral by introducing a scalar product (for real-valued classical fields)

$$(\phi_k, \phi_j) = \int_0^1 \int_{\mathbb{R} \times [0,l]^{d-1}} \phi_j \phi_k d\overrightarrow{x'}\, dt' \qquad (3.2.20)$$

and an operator

$$L = -\frac{iTa^d}{2\pi\hbar r^2} \left(-\frac{\rho}{T^2} \frac{\partial^2}{\partial t'^2} + \sum_{n=1}^d \frac{G_n}{a^2} \frac{\partial^2}{\partial x_n'^2} - V''(\varphi) \right) \qquad (3.2.21)$$

with r^2 as normalising factor introduced to eliminate (ϕ_j, ϕ_j) from calculations and explained in detail further in the text

$$e^{\frac{i}{\hbar}S(\varphi)} \int_{\mathbb{R}} e^{-r^2\pi \sum_{j,k}(\phi_k, L\phi_j)w_j w_k} \prod_f dw_f. \qquad (3.2.22)$$

Further transformations rely on the form of the potential and the classical solution taken. If we consider a solution dependent on a single variable (including one obtained by Lorentz transform or rotation), then operator L is a sum of one-dimensional symmetric (for our definition of scalar product) second order differential operators, which means its eigenfunctions form an orthogonal base provided that the border conditions do not couple the single variable of importance with any other. In such a case we can use this base to simplify the propagator into a product of standard Gaussian integrals

$$e^{\frac{i}{\hbar}S(\varphi)} \int_{\mathbb{R}} e^{-\pi \sum_j \lambda_j w_j^2} \prod_f dw_f, \qquad (3.2.23)$$

which can be easily solved

$$e^{\frac{i}{\hbar}S(\varphi)} \prod_j (\lambda_j)^{-\frac{1}{2}}. \qquad (3.2.24)$$

Finally, the product of eigenvalues of L can be represented simply as a determinant of L

$$\frac{e^{\frac{i}{\hbar}S(\varphi)}}{\sqrt{\det[L]}}, \qquad (3.2.25)$$

which is the main point of [61] (which also provides proper regularization). Up to this point we have neglected the left hand side of (3.1.6). In case of a static classical field (which also implies $\forall_T \psi_T = \psi_0$ and $S(\varphi) = -TE_c$) it can be approximated by

$$\langle \psi_0 | e^{-\frac{i}{\hbar}TH} | \psi_0 \rangle \approx e^{-\frac{i}{\hbar}TE_q} \langle \psi_0 | \psi_0 \rangle, \qquad (3.2.26)$$

where E_q is the total energy of a given field [101]. Earlier publications often used the notion of effective Lagrangian or effective action to obtain the same results. However, the connection to the quantum eigenvalue problem might be a useful notion in quantization of non-static classical fields. By combining both approximations one obtains

$$E_q = E_c + \frac{i\hbar}{2T} \ln(\det[L]) + \frac{iT}{\hbar} \ln \langle \psi_0 | \psi_0 \rangle, \qquad (3.2.27)$$

or more precisely

$$E_q = E_c + \Re \left(\frac{i\hbar}{2T} \ln \det[L] \right). \qquad (3.2.28)$$

This expression needs to be regularized, considering that the spectrum of any differential operator is unbounded. We will describe the chosen regularization scheme in the next subsection.

3.3   Generalized zeta-function regularization

Chosen regularization scheme consists of two steps. First is based on the assumption, that for a minimal energy solution (vacuum state) the corrections should vanish [61]. Therefore

$$E_q = E_c + \Re \left[\frac{i\hbar}{2T} (\ln \det[L] - \ln \det[L_0]) \right], \qquad (3.3.29)$$

where

$$L_0 = -\frac{iTa^d}{2\pi\hbar r^2}\left(-\frac{\rho}{T^2}\frac{\partial^2}{\partial t'^2} + \sum_{n=1}^{d}\frac{G_n}{a^2}\frac{\partial^2}{\partial x_n'^2} - C\right) \tag{3.3.30}$$

with $C = const$ dependent on the particular potential V chosen. This also assumes, that the potential V is a function of ϕ only. At this point it is necessary to rewrite the determinants of both operators in a form, which would allow the subtraction. For this purpose we use a well known generalized zeta-function [103, 104]

$$\zeta_L(s) = \sum_j \lambda_j^{-s}, \tag{3.3.31}$$

where λ_j are eigenvalues of L. As can be seen

$$\ln\det[L] = -\frac{\partial\zeta_L}{\partial s}(0). \tag{3.3.32}$$

The zeta-function itself will be constructed through a set of transformations on a heat equation Green function as was shown in [72]

$$\left(\frac{\partial}{\partial\tau} + L\right)g_L\left(\tau, t', t_0', \overrightarrow{x'}, \overrightarrow{x'}_0\right) = \delta(\tau)\delta(t' - t_0')\delta\left(\overrightarrow{x'} - \overrightarrow{x'}_0\right) \tag{3.3.33}$$

with border conditions on g_L the same as for the functions in Maslov representation and an additional condition

$$\forall_{\tau<0}\ g_L\left(\tau, t', t_0', \overrightarrow{x'}, \overrightarrow{x'}_0\right) \equiv 0. \tag{3.3.34}$$

In such a situation the Green function can be expressed as a formal sum over eigenstates of L

$$g_L\left(\tau, t', t_0', \overrightarrow{x'}, \overrightarrow{x'}_0\right) = \sum_j e^{-\lambda_j\tau}\frac{\phi_j\left(t', \overrightarrow{x'}\right)\phi_j\left(t_0', \overrightarrow{x'}_0\right)}{(\phi_j, \phi_j)}\Theta(\tau), \tag{3.3.35}$$

where Θ denotes Heaviside step function. This definition of the Green function relies on the expression of Dirac delta function through a formal sum

$$\delta(t' - t_0')\delta\left(\overrightarrow{x'} - \overrightarrow{x'}_0\right) = \sum_j \frac{\phi_j\left(t', \overrightarrow{x'}\right)\phi_j\left(t_0', \overrightarrow{x'}_0\right)}{(\phi_j, \phi_j)} \tag{3.3.36}$$

over all elements of a orthogonal base. After inserting (3.3.35) into (3.3.33) we obtain

$$\delta(\tau)\sum_j e^{-\lambda_j\tau}\frac{\phi_j\left(t', \overrightarrow{x'}\right)\phi_j\left(t_0', \overrightarrow{x'}_0\right)}{(\phi_j, \phi_j)} = \delta(\tau)\delta(t' - t_0')\delta\left(\overrightarrow{x'} - \overrightarrow{x'}_0\right). \tag{3.3.37}$$

As can be seen the formal sum on the left side of equation will converge to the definition of δ function for $\tau \to 0$. We can build the renormalized zeta-function by following transformations:

$$\begin{aligned}\zeta(s) &= \frac{1}{\Gamma(s)}\int_0^\infty \tau^{s-1}\int_0^1\int \mathbb{R}\times[0, al]^{d-1}\left[g_L\left(\tau, t', t', \overrightarrow{x'}, \overrightarrow{x'}\right)\right.\\&\quad\left. - g_{L_0}\left(\tau, t', t', \overrightarrow{x'}, \overrightarrow{x'}\right)\right]d\overrightarrow{x'}\,dt'\,d\tau.\end{aligned} \tag{3.3.38}$$

This method of constructing the generalized zeta-function allows for an easy introduction of additional variables thanks to the property of heat equation Green function [72]

$$g_{L_1+L_2} = g_{L_1} g_{L_2} \tag{3.3.39}$$

up to a factor of $\Theta(\tau)$ if L_1 and L_2 depend on different, independent variables. Since usually one calculates the Green function for $\tau > 0$ and adds the step function later, it's not much of a setback. To explain this property, let us assume $\phi_{1,j}(x)$ as eigenfunctions of L_1 with eigenvalues $\lambda_{1,j}$ and $\phi_{2,n}(y)$ as eigenfunctions of L_2 with eigenvalues $\lambda_{2,n}$. Considering the aforementioned independence of variables $\phi_{1,j}(x)\phi_{2,n}(y)$ are eigenfunctions of $L_1 + L_2$ with eigenvalues $\lambda_{1,j} + \lambda_{2,n}$. Knowing this we can build

$$g_{L_1+L_2} = \sum_{j,n} e^{-(\lambda_{1,j}+\lambda_{2,n})\tau} \frac{\phi_{1,j}(x)\phi_{2,n}(y)\phi_{1,j}(x_0)\phi_{2,n}(y_0)}{(\phi_{1,j}\phi_{2,n}, \phi_{1,j}\phi_{2,n})} \Theta(\tau). \tag{3.3.40}$$

Considering the scalar product we use, we can write

$$(\phi_{1,j}\phi_{2,n}, \phi_{1,j}\phi_{2,n}) = (\phi_{1,j}, \phi_{1,j})_1 (\phi_{2,n}, \phi_{2,n})_2. \tag{3.3.41}$$

From here we obtain

$$g_{L_1+L_2} = \sum_{j} e^{-\lambda_{1,j}\tau} \frac{\phi_{1,j}(x)\phi_{1,j}(x_0)}{(\phi_{1,j}, \phi_{1,j})_1} \sum_{n} e^{-\lambda_{2,n}\tau} \frac{\phi_{2,n}(y)\phi_{2,n}(y_0)}{(\phi_{2,n}, \phi_{2,n})_2} \Theta(\tau), \tag{3.3.42}$$

which brings us back to (3.3.39). It is convenient to introduce an additional function

$$\gamma_L(\tau) = \int_0^1 \int g_L(\tau, t', t', \overrightarrow{x'}, \overrightarrow{x'}) d\overrightarrow{x'} dt', \tag{3.3.43}$$

for which the (3.3.39) relation holds as well. For the purpose of this thesis we consider only one-dimensional classical solutions and use a few additional definition

$$L_1 = A\left(\frac{\partial^2}{\partial x_1'^2} - \frac{a^2 V''(\varphi(x_1'))}{G}\right), \tag{3.3.44}$$

$$A = -\frac{iTGa^{d-2}}{2\pi\hbar r^2} \tag{3.3.45}$$

with the exact form of potential V and solution φ specified in Chapter 5. It also means we can restrict L_0 to the x variable only. For the rest of the publication x_1' will be denoted as x and $\forall_n G_n = G$. Furthermore

$$L_2 = -\frac{A}{c^2}\frac{\partial^2}{\partial t'^2}, \tag{3.3.46}$$

$$L_3 = A\sum_{n\neq 1}\frac{\partial^2}{\partial x_n'^2} \tag{3.3.47}$$

with a connection to (3.2.21) by $\forall_n G_n = G$ and $c^2 = \frac{GT^2}{Ma^2}$. Considering the simple form of L_2 and L_3 we can readily calculate (assuming a span of l for all x_n' and

continuum approximation of spectra)

$$\gamma_{L_2} = \sqrt{\frac{c^2}{4\pi A\tau}}, \tag{3.3.48}$$

$$\gamma_{L_3} = \left(-\frac{l}{4\pi A\tau}\right)^{\frac{d-1}{2}}. \tag{3.3.49}$$

We can also transform the equation for g_{L_1} to extract A out of it with a simple substitution $\tau = \frac{\tau_A}{|A|}$ (due to the nature of Dirac delta function it is better not to scale it by an imaginary constant). We will get

$$\left\{\frac{\partial}{\partial \tau_A} - i\left[\frac{\partial^2}{\partial x^2} - \frac{a^2 V''(\varphi(x))}{G}\right]\right\} g_{L_1}(\tau_A, x, x_0) = \delta(\tau_A)\delta(x - x_0). \tag{3.3.50}$$

With such notation we obtain a final form of zeta-function

$$\zeta(s) = \frac{icl^{\frac{d-1}{2}}\pi^{\frac{d}{2}}}{2^d|A|^{-s}\Gamma(s)} \int_0^\infty \tau_A^{s-\frac{d+2}{2}} \int \mathbb{R} \left[g_{L_1}(\tau_A, x, x) - g_{L_0}(\tau_A, x, x)\right] dx d\tau_A \tag{3.3.51}$$

and the final form for energy corrections

$$\begin{aligned}
\Delta E &= \Re\left\{\lim_{s\to 0}\frac{\partial}{\partial s}\frac{\hbar cl^{\frac{d-1}{2}}\pi^{\frac{d}{2}}}{2^{d+1}T|A|^{-s}\Gamma(s)}\int_0^\infty \tau_A^{s-\frac{d+2}{2}}\right. \\
&\quad \left.\times \int \left[g_{L_1}(\tau_A, x, x) - g_{L_0}(\tau_A, x, x)\right] dx d\tau_A\right\}.
\end{aligned} \tag{3.3.52}$$

At this point the renormalising factor r^2 (often referred to as the mass scale [72]) is chosen to cut out all logarithmically divergent terms arising from differentiation of A^{-s} and possibly the Mellin integral as well. It is important to stress, that the choice of its value has impact on quantitative results and is not necessarily obvious. On purely mathematical level r^2 can be viewed as a free parameter of the theory (which is a reason for keeping it unspecified in works of Konoplich [72]), but it seems so only because we usually are unable to construct the whole propagator, which would allow us to use the normalising condition of the propagation operator

$$\forall_{0<T'<T} \langle\psi_T| e^{-\frac{i}{\hbar}TH} |\psi_0\rangle = \sum_{\psi_{T'}} \langle\psi_T| e^{-\frac{i}{\hbar}(T-T')H} |\psi_{T'}\rangle \langle\psi_{T'}| e^{-\frac{i}{\hbar}T'H} |\psi_0\rangle \tag{3.3.53}$$

to properly set the value of r^2 (see appendix A for a sample use of the same method in the case of harmonic oscillator). Yet, to obtain physically relevant results one has to find a way of estimating the normalising factor. This problem will be further discussed in Sections 5.1 and 5.2.1. Similar issues might arise with the L_0 for fields spanning over a finite domain, where any choice of the constant potential leads to finite results. However, known results for fields spanning over infinite domains suggest, that the lowest eigenvalue of L_0 should coincide with start of the unbound states band.

Chapter 4

Green function diagonal equation

4.1 Derivation

Let us consider a heat kernel problem

$$\left(\frac{\partial}{\partial \tau_A} + \frac{\partial^2}{\partial x^2} - U(x)\right) g(x, x_0, \tau_A) = \delta(x - x_0)\delta(\tau_A), \qquad (4.1.1)$$

where $g(x, x_0, \tau_A) \in S$ is the fundamental solution over Schwartz space S and $\delta(x - x_0)$, $\delta(\tau_A)$ are Dirac delta-functions. After Laplace transformation, an ODE, parameterized by p, appears

$$\left(p + \frac{\partial^2}{\partial x^2} - U(x)\right) \hat{g}(x, x_0, p) = \delta(x - x_0). \qquad (4.1.2)$$

The construction of generalized zeta-function in fact relies upon the Green function diagonal [70, 72, 103]. In [103] a statement about $\hat{g}(p, x, x) = G(p, x)$, is used. Namely, $G(p, x)$ solves the equation

$$2GG''_{xx} - (G'_x)^2 - 4(U(x) - p)G^2 + 1 = 0 \qquad (4.1.3)$$

on condition, that $U(x)$ is bounded [103]. As can be seen, this is a form of Drach equation [105] (main points of this and related works were described in [106]) used in a different context. The equation resembles one of derived by Hermit for the Lamè equation [107], see also [108]. We will now explicitly prove the connection between heat kernel problem and Drach equation.

Thesis: If $\hat{g}(p, x, x_0)$ is a solution of (4.1.2), then $\hat{g}(p, x, x)$ solves (4.1.3).

Proof: Let us consider homogeneous equation

$$\left(p + \frac{\partial^2}{\partial x^2} - U(x)\right) f(p, x, x_0) = 0. \qquad (4.1.4)$$

The fundamental solution of (4.1.4) is built by standard procedure. It has two linearly independent solutions, for example ϕ and ψ, decaying respectively at

$-\infty$ and $+\infty$. One can represent $\hat{g}$ through ϕ and ψ either for $x < x_0$, or $x > x_0$ with a sewing condition determined by equation (4.1.2)

$$\hat{g}_D(p, x, x_0) = \begin{cases} A(x_0)\phi(p, x), & x \leqslant x_0 \\ B(x_0)\psi(p, x), & x \geqslant x_0. \end{cases} \qquad (4.1.5)$$

From the continuity condition of $\hat{g}$ one obtains

$$A(x_0)\phi(p, x_0) = B(x_0)\psi(p, x_0), \qquad (4.1.6)$$

which leads to:

$$\begin{aligned} A(x_0) &= C(x_0)\psi(p, x_0), & (4.1.7) \\ B(x_0) &= C(x_0)\phi(p, x_0). & (4.1.8) \end{aligned}$$

Due to the symmetry of Green function in respect to exchange of x and x_0, $C(x_0)$ is constant (later referred as C). To obtain the condition for derivatives of ϕ and ψ one integrates (4.1.2) with respect to x in an ε neighbourhood of x_0

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} \left(p + \frac{\partial^2}{\partial x^2} - U(x)\right) \hat{g}(p, x, x_0)\, dx = 1, \qquad (4.1.9)$$

$$\frac{\partial \hat{g}_D}{\partial x}(p, x, x_0)\Bigg|_{x=x_0-\varepsilon}^{x_0+\varepsilon} + \int_{x_0-\varepsilon}^{x_0+\varepsilon} (p - U(x))\, \hat{g}(p, x, x_0)\, dx = 1 \qquad (4.1.10)$$

and insert the proper form of g_D

$$\frac{\partial \phi}{\partial x}(p, x_0+\varepsilon)\, C\psi(p, x_0) - \frac{\partial \psi}{\partial x}(p, x_0-\varepsilon)\, C\phi(p, x_0) + \int_{x_0-\varepsilon}^{x_0+\varepsilon} (p - U(x))\, \hat{g}(p, x, x_0)\, dx = 1. \qquad (4.1.11)$$

In $\varepsilon \to 0$ limit above equation reduces to

$$\frac{\partial \phi}{\partial x}(p, x_0)\, C\psi(p, x_0) - \frac{\partial \psi}{\partial x}(p, x_0)\, C\phi(p, x_0) = 1. \qquad (4.1.12)$$

Since solutions of (4.1.4) are linear, one can assume $C = 1$. Therefore (4.1.12) reduces to

$$\frac{\partial \phi}{\partial x}(p, x_0)\, \psi(p, x_0) = \frac{\partial \psi}{\partial x}(p, x_0)\, \phi(p, x_0) + 1. \qquad (4.1.13)$$

Actual proof will be made, by inserting (4.1.5) to (4.1.3). For brevity function arguments will be omitted and prime will denote a derivative with respect to x

$$2\psi\phi\left(\psi''\phi + 2\psi'\phi' + \psi\phi''\right) - \left(\psi'\phi + \psi\phi'\right)^2 - 4(U(x) - p)\psi^2\phi^2 + 1 = 0. \qquad (4.1.14)$$

We arrange the elements of the equation

$$2\psi^2\phi\left(\phi'' - (U(x) - p)\phi\right) + 2\psi\phi^2\left(\psi'' - (U(x) - p)\psi\right) + 4\psi'\phi'\psi\phi - \left(\psi'\phi + \psi\phi'\right)^2 + 1 = 0 \qquad (4.1.15)$$

to use (4.1.4) in order to nullify the two first elements. One also uses property (4.1.13):

$$4\psi'\phi'\psi\phi - (2\psi'\phi + 1)^2 + 1 = 0, \tag{4.1.16}$$
$$4\psi'\phi'\psi\phi - 4\psi'^2\phi^2 - 4\psi'\phi - 1 + 1 = 0, \tag{4.1.17}$$
$$\psi'\phi'\psi\phi - \psi'^2\phi^2 - \psi'\phi = 0, \tag{4.1.18}$$
$$\psi'^2\phi^2 + \psi'\phi - \psi'^2\phi^2 - \psi'\phi = 0. \tag{4.1.19}$$

Thus the proof is concluded. It is important to note, that the transition is general and doesn't rely on the nature of $U(x)$ as long as it is bound, but its usefulness is dependent on a few qualities of the potential though.

4.2 General properties

It is most useful, if there exists a variable transition $x \to z$, $U(x) \to u(z)$, in which the solution can be written in a form

$$G(p, z) = \frac{P(p, z)}{2\sqrt{Q(p)}}, \tag{4.2.20}$$

$$P(p, z) = \sum_{n=0}^{N} p^n \sum_{l=0}^{M_n} P_{n,l} z^l, \tag{4.2.21}$$

$$Q(p) = \sum_{n=0}^{2N+1} q_n p^n. \tag{4.2.22}$$

Basic conditions for it to be possible are:

1. u is a polynomial in z,

2. $(z'_x)^2$ is a polynomial in z (note, that $z''_{xx} = \frac{1}{2}\frac{\partial}{\partial z}(z'_x)^2$).

This does not ensure simplicity of solutions as will be shown further in the text. At this point, it is important to notice, that the second condition restricts $z(x)$ apart from a class of elementary functions to elliptic and hyperelliptic functions - see, e.g. [106]. It is also very interesting to see, that the conditions are very similar to those of a Lame operator - those similarities will be further reinforced after following analysis.

Let us assume, that $(z'_x)^2$ and u are polynomials in the variable z of order $L+1$, and K respectively, hence z''_{xx} is a polynomial of the order L. We also assume, that coefficient by the highest power term of $(z'_x)^2$ is equal to 1 (which is always attainable through scaling). After the change of variables, the equation will take the form:

$$2P(P''(z'_x)^2 + P'z''_{xx}) - (P'z'_x)^2 - 4(u(z) - p)P^2 + 4Q = 0. \tag{4.2.23}$$

We will now proceed to analyse the solution by separating the equation in respect to powers of p and z. The equation for $p^0, z^{2M_0+max(K,L-1)}$ takes following form:

- if $K > L - 1$,

$$4u_K P_{0,M_0} = 0; \qquad (4.2.24)$$

- if $K \leqslant L - 1 \quad \wedge \quad M_0 \geqslant 1$,

$$2P_{0,M_0}^2[M_0(M_0 - 1) + \frac{L+1}{2}M_0] - P_{0,M_0}^2 M_0^2 - 4u_K P_{0,M_0}^2 \delta_{K,L-1} = 0, \quad (4.2.25)$$

which reduces to

$$M_0^2 + (L - 1)M_0 - 4u_K \delta_{K,L-1} = 0. \qquad (4.2.26)$$

This equation gives us a condition on the amplitude of the potential for which polynomial expansion of the solution is possible (assuming $K = L - 1$)

$$u_K = \frac{M_0(M_0 + L - 1)}{4}. \qquad (4.2.27)$$

Note, that $M_0 = 0$ leads to $K = 0$ (this case will be examined later in the text). Another conclusion is, that $K \leqslant L - 1$ is necessary for sought type of solutions. Furthermore, this leads to following, more precise conditions:

$$\begin{cases} L & \geqslant & 1 & \text{(due to } K \geqslant 0), \\ K & = & L - 1 & \text{(for } M_0 \text{ to have positive value).} \end{cases} \qquad (4.2.28)$$

It is very interesting to see, that for $K = 1$ we obtain exactly the conditions for a Lame operator. Now we analyse the equations for specific powers of p starting from the highest (p^{2N+1})

$$4 \left(\sum_{l=0}^{M_N} P_{N,l} z^l \right)^2 + 4q_{2N+1} = 0, \qquad (4.2.29)$$

which leads to following conclusions:

$$M_N = 0 \quad \wedge \quad P_{N,0}^2 = -q_{2N+1}. \qquad (4.2.30)$$

Let us now look at subsequent equations for descending powers of p. For p^{2N} we have

$$-4u(z)P_{N,0}^2 + 8P_{N,0} \sum_{l=0}^{M_{N-1}} P_{N-1,l} z^l + 4q_{2N} = 0, \qquad (4.2.31)$$

which leads to

$$\sum_{l=0}^{M_{N-1}} P_{N-1,l} z^l = \frac{1}{2}\left[P_{N,0}u(z) - \frac{q_{2N}}{P_{N,0}} \right] \qquad (4.2.32)$$

and in consequence

$$M_{N-1} = K. \qquad (4.2.33)$$

For p^{2N-1} we get (for the highest power of z):

- on condition $M_{N-2} > M_{N-1} + K$ ($z \neq x^2$)

$$4P_{N,0}P_{N-2,M_{N-2}} = 0, \qquad (4.2.34)$$

- on condition $M_{N-2} \leqslant M_{N-1} + K$

$$2P_{N,0}P_{N-1,M_{N-1}}[M_{N-1}(M_{N-1} - 1) + \frac{K+2}{2}M_{N-1}]$$
$$-2P_{N,0}P_{N-1,M_{N-1}} - 8u_K P_{N,0}P_{N-1,M_{N-1}}$$
$$+8P_{N,0}P_{N-2,M_{N-2}}\delta_{M_{N-2},L-1} + 4q_{2N-1} = 0. \quad (4.2.35)$$

It is obvious, that $M_{N-2} \leqslant M_{N-1} + K$ is a necessary condition for (4.2.20). Now we can consider a general rule for all remaining equations.

Thesis: $\forall_{0 \leqslant k < N-1} M_k \leqslant (N-k)K$.

Proof by induction:

- If $M_k > (N-k)K$, then equation for p^{N+k+1} and highest power of z takes form:

$$4P_{N,0}P_{k,M_k} = 0. \qquad (4.2.36)$$

- If $M_k \leqslant (N-k)K$ and $\forall_{k < l < N} M_l = (N-l)K$ (possibility giving the highest possible value of M_k), then equation for p^{N+k+1} and highest power of z takes form:

$$\begin{aligned}
0 = {} & 2P_{N,0}P_{k+1}[M_{k+1}(M_{k+1} - 1) + \frac{K+2}{2}M_{k+1}] \\
& + 2\sum_{n=k+2}^{N-1} P_{n,M_n}P_{N-n+k+1,M_{N-n+k+1}}[M_{N-n+k+1}(M_{N-n+k+1} - 1) \\
& + \frac{K+2}{2}M_{N-n+k+1}] - 2\sum_{n=k+2}^{N-1} P_{n,M_n}P_{N-n+k+1}M_{N-n+k+1}M_n \\
& - 4u_K \sum_{n=k+1}^{N} P_{n,M_n}P_{N-n+k+1} + 4\sum_{n=k+1}^{N-1} P_{n,M_n}P_{N-n+k,M_{N-n+k}} \\
& + 4\delta_{M_k,M_{k+1}+L-1}P_{N,0}P_{k,M_k}. \qquad (4.2.37)
\end{aligned}$$

Thus the solution exists only if the thesis holds. This leads directly to a minimal condition on N:

$$N \geqslant \frac{M_0}{K} \qquad \forall M_0 \geqslant K. \qquad (4.2.38)$$

Summarising, existence of solutions of form (4.2.20) depends on the power of the potential (K), power of $(z'_x)^2$ ($L \leqslant 1$ can give abnormal results), amplitude of the highest power term of the potential and there exists a definite formula for the minimal value of N.

4.3   Uniqueness of solutions

Let us assume according to conclusions of the previous section

$$(z'_x)^2 = z^{K+2} + \sum_{n=0}^{K+1} c_n z^n \tag{4.3.39}$$

and a solution in form of

$$P = \sum_{l=0}^{M} P_l(p) z^l, \tag{4.3.40}$$

$$Q = \sum_{k=0}^{2N+1} q_k p^k \tag{4.3.41}$$

and separate the equation in respect to powers of z starting with the highest power (z^{2M+K}) assuming $K > 0$

$$P_M^2 \left[2M(M-1) + M(K+2) - M^2 - 4u_K \right] = 0, \tag{4.3.42}$$

which simplifies to

$$P_M^2(M^2 + MK - 4u_K) = 0. \tag{4.3.43}$$

Considering $K \geqslant 0$ the above equation has at most one positive solution. Equation for z^{2M+K-1}

$$\begin{aligned}
0 = \; & P_M P_{M-1}[2M^2 + MK + 2(M-1)^2 + (M-1)K - 2M(M-1) - 8u_K] \\
& + P_M^2[2M(M-1)c_{K+1} + M(K+2)c_{K+1} - M^2 c_{K+1} - 4u_{K-1} + 4p\delta_{K,1}]
\end{aligned} \tag{4.3.44}$$

can be divided by P_M and give

$$\begin{aligned}
0 = \; & P_{M-1}[2M^2 + 2M(K-1) - K - 8u_K + 2] \\
& + P_M[2M(M-1)c_{K+1} + M(K+2)c_{K+1} - M^2 c_{K+1} - 4u_{K-1} + 4p\delta_{K,1}].
\end{aligned} \tag{4.3.45}$$

By substituting u_K from (4.3.43) we obtain

$$2P_{M-1}(2 - K - 2M) + P_M(4u_K c_{K+1} - 4u_{K-1} + 4p\delta_{K,1}) = 0. \tag{4.3.46}$$

Considering that M and K are either both equal to 0 or both positive integers (as seen in [109]) $2 - K - 2M \neq 0$, so P_{M-1} is of one power higher in p than P_M if and only if the potential is linear and it is of the same power in any other situation. More importantly, P_{M-1} is unambiguously determined by P_M.

Theorem: All P_l are linearly proportional to P_M.

Proof by induction:

Let us consider the equation for z^{M+K+l} for $0 \leqslant l < M - 1$ and assume the theorem holds for all $l < k < M$

$$
\begin{aligned}
P_l P_M [2l(l-1) + l(K+2) + 2M(M-1) \\
+ M(K+2) - 2Ml - 8u_K] + P_M^2 F(U, z'^2) &= 0,
\end{aligned}
\tag{4.3.47}
$$

where F contains all parts of equation not written explicitly. For the theorem to hold, it suffices to prove, that $2l(l-1) + l(K+2) + 2M(M-1) + M(K+2) - 2Ml - 8u_K \neq 0$. Let us consider the opposite:

$$
2l(l-1) + l(K+2) + 2M(M-1) + M(K+2) - 2Ml - 8u_K = 0.
\tag{4.3.48}
$$

After substitution of u_K we obtain

$$
2l(l-1) + l(K+2) + 2M(M-1) + M(K+2) - 2Ml - 2M^2 - 2MK = 0,
\tag{4.3.49}
$$

which can be simplified

$$
\begin{aligned}
2l^2 + lK + 2M^2 + MK - 2Ml - 2M^2 - 2MK &= 0, \tag{4.3.50} \\
2l^2 + lK - MK - 2Ml &= 0, \tag{4.3.51} \\
(2l + K)(l - M) &= 0. \tag{4.3.52}
\end{aligned}
$$

This equation would only be true for $l = M$, which ends the proof. There are two important conclusions from this theorem:

1. P_M is a free parameter of the solution, that doesn't have any meaning, since $P \propto P_M$ and $Q \propto P_M^2$.

2. Solutions of form (4.2.20) are unique up to a sign (this uncertainty is due to the quadratic nature of the equation and not just due to the square root in the chosen form) and can only exist, if there exists a natural M solving equation (4.3.43).

We will now proceed to analyse the $K = 0$ case. In [109] it was established, that a solution for constant potential does not depend on z and P can be a polynomial of arbitrary power. We will now show, that all those solutions are equal by explicitly solving the equation

$$
-4(u - p)P(p)^2 + 4Q(p) = 0,
\tag{4.3.53}
$$

which immediately gives

$$
Q(p) = (u - p)P(p)^2,
\tag{4.3.54}
$$

where P can be an arbitrary function of p. This gives

$$
G(p, z) = \frac{P(p)}{2\sqrt{P^2(p)(u - p)}} = \frac{1}{2\sqrt{u - p}}.
\tag{4.3.55}
$$

4.4 Solving algorithms

4.4.1 Polynomial expansion in $z(x)$

Following the proof of uniqueness we can readily construct the solution of (4.1.3) by iteratively solving the equations for descending powers of z with respect to $P_l(p)$. There is no explicit method of finding the proper change of variables, which would fulfil the necessary conditions (4.2), so $z(x)$ and preferably $(z'_x)^2$ as well as $u(z)$ have to be given at the start. In first step one calculates the value of M from (4.3.43). For implementation in programing languages capable of symbolic computation it is convenient to write the whole left hand side of the equation (with P, $u(z)$, $(z'_x)^2$ and z''_{xx} in explicit polynomial form in z) as a single function (for example $F(p, z)$). We can then proceed to extract equations for specific powers of z by differentiation in z and taking a limit of $z \to 0$. For example

$$\lim_{z \to 0} \frac{\partial^n}{\partial z^n} F(p, z) = 0 \qquad (4.4.56)$$

is the equation for z^{2M+K-n}. As was shown in the previous subsection, we can assume $P_M(p) = 1$. From equations for z^{2M+K-1} to z^{M+K} we will obtain all $P_n(p)$ from $P_{M-1}(p)$ to $P_0(p)$ respectively. Equations from z^{M+K-1} to z^1 will be used to validate the solution and the equation for z^0 will be used to calculate $Q(p)$ thus ending the process. Sample implementation in *Mathematica* can be found in Appendix B.

4.4.2 Polynomial expansion in p

The equation can still be solved, even if P is only a polynomial in p. We assume

$$
\begin{aligned}
P &= \sum_{l=0}^{N} P_l(x) p^l, \\
Q &= \sum_{l=0}^{2N+1} q_l p^l.
\end{aligned}
\qquad (4.4.57)
$$

We can obtain all P_l up to a constant in an iterative procedure following the analysis done in Section 4.2. For p^{2N+1}

$$P_N = -q_{2N+1}. \qquad (4.4.58)$$

For p^{2N}

$$P_{N-1} = \frac{P_N}{2} u(x) - \frac{q_{2N}}{2P_N}. \qquad (4.4.59)$$

For p^{N+k} (where k is a positive integer)

$$
\begin{aligned}
P_{k-1} = \;& \frac{1}{8P_N} \left\{ 4q_{N+k} + \sum_{l=k}^{N} \left[2P_l \left(P''_{N-l+k} - 2P_{N-l+k} u(x) \right) \right. \right. \\
& \left. \left. - P'_l P'_{N-l+k} \right] + 4 \sum_{l=k}^{N-1} P_l P_{N-l+k-1} \right\}.
\end{aligned}
\qquad (4.4.60)
$$

Using this method we can obtain all P_l, but there remains the problem of obtaining all q_l. They can be obtained from the remaining equations, but as yet there is no explicit method unless the conditions of (4.4.1) are met. As of yet there is also no method of checking, if a solution of form (4.4.57) exists and what would be the proper value of N in such a situation. Those problems could probably be solved, if one could transform the recurrent equation for P_k into one relying on $u(x)$, q_l and P_N only.

4.5 Generalizations

4.5.1 Scaling

The most important aspect, that needs to be incorporated is the scaling of operator $\frac{\partial^2}{\partial x^2} - U(x)$ through a complex constant. Let us consider an equation

$$\left[p' + C \left(\frac{\partial^2}{\partial x^2} - U(x) \right) \right] \hat{g}'_{L_1}(p', x, x_0) = \delta(x - x_0) \tag{4.5.61}$$

with C as a complex constant. If we were to substitute $p' = Cp$

$$C \left[p + \frac{\partial^2}{\partial x^2} - U(x) \right] \hat{g}'_{L_1}(Cp, x, x_0) = \delta(x - x_0), \tag{4.5.62}$$

then it's easy to see, that $C\hat{g}'_{L_1}(Cp, x, x)$ solves the equation (4.1.3). Therefore, if $G(p, x)$ is a solution of (4.1.3), then $\frac{1}{C} G\left(\frac{p'}{C}, x \right)$ is the diagonal of the Green function defined by (4.5.61).

4.5.2 Application to other differential equations

Considering, that the Drach equation does not contain derivatives in respect to p, we can substitute this variable with an arbitrary function of a number of parameters as long as they are independent of x

$$p = f(p_1, \ldots, p_n). \tag{4.5.63}$$

Considering the connection between the Drach equation and the original heat kernel problem (4.1.1), it means, that we can potentially calculate Green function diagonals for a very wide range of differential equations. For instance, substitution

$$p = p_1^2 \tag{4.5.64}$$

would change the initial problem (4.1.1) into an elliptic equation. The main caveat is, such substitutions will inevitably complicate the inverse Laplace transform - especially if we introduce more parameters. This can make it impossible to extract explicit results.

Chapter 5

Quantum corrections to energy

5.1 General remarks

All potentials considered in following sections have many similarities, so it is useful to study general properties of a family of potentials of form

$$u(x) = b^2 f(bx) + C, \tag{5.1.1}$$

where b and C are real constants and f is an arbitrary integrable function. Our goal will be to extract as much information from (3.3.52) as possible without explicitly solving the Green function problem. This will be most useful for periodic and quasi-periodic solutions of Sine-Gordon and ϕ^4 models, where exact analytic solutions are difficult to obtain. Let us begin with the Green function equation for L_1 with (5.1.1) as the potential

$$\left[\frac{\partial}{\partial \tau} + A \left(\frac{\partial^2}{\partial x^2} - b^2 f(bx) - C \right) \right] g_{L_1}(\tau, x, x_0) = \delta(\tau)\delta(x - x_0). \tag{5.1.2}$$

We will now try to remove b from the equation by a series of substitutions. Firstly, we rescale the x variable

$$x_b = bx, \tag{5.1.3}$$

which will give us

$$\left[\frac{\partial}{\partial \tau} + Ab^2 \left(\frac{\partial^2}{\partial x_b^2} - f(x_b) - \frac{C}{b^2} \right) \right] g_{L_1}(\tau, x_b, x_{b,0}) = b\delta(\tau)\delta(x_b - x_{b,0}). \tag{5.1.4}$$

Next we rescale τ similarly as in Section 3.3

$$\tau = \frac{\tau_b}{|A|b^2} \tag{5.1.5}$$

obtaining

$$\left[\frac{\partial}{\partial \tau_b} - i \left(\frac{\partial^2}{\partial x_b^2} - f(x_b) - \frac{C}{b^2} \right) \right] g_{L_1}(\tau_b, x_b, x_{b,0}) = b\delta(\tau_b)\delta(x_b - x_{b,0}). \tag{5.1.6}$$

From here we can extract both b and C by rescaling the Green function

$$g_{L_1}(\tau_b, x_b, x_{b,0}) = b e^{\frac{C}{b^2}\tau_b} g_{L_1,b}(\tau_b, x_b, x_{b,0}). \tag{5.1.7}$$

After inserting that form into (5.1.6), we find that $g_{L_1,b}(\tau_b, x_b, x_{b,0})$ solves the equation

$$\left[\frac{\partial}{\partial\tau_b} - i\left(\frac{\partial^2}{\partial x_b^2} - f(x_b)\right)\right] g_{L_1,b}(\tau_b, x_b, x_{b,0}) = \delta(\tau_b)\delta(x_b - x_{b,0}). \tag{5.1.8}$$

The same procedure has to be done for the Green function of operator L_0 - both rescaling of the x and τ variables as well as potential shift by $\frac{C}{b^2}$ (if done after rescaling of τ) even if the constant potential in L_0 has a different value than $\frac{C}{b^2}$. As for the Green functions (or γ functions (3.3.43)) of L_2 and L_3 operators, it is most convenient to express them through τ_b as well

$$\gamma_{L_2}(\tau_b) = \sqrt{\frac{ic^2b^2}{4\pi\tau_b}}, \tag{5.1.9}$$

$$\gamma_{L_3}(\tau_b) = \left(\frac{l^2b^2}{i4\pi\tau_b}\right)^{\frac{d-1}{2}}. \tag{5.1.10}$$

With this information we can reproduce the equation for energy corrections remembering to change the integration over x to integration over x_b

$$\Delta E = \Re\left\{\lim_{s\to 0}\frac{\partial}{\partial s}\frac{\hbar c b^d l^{d-1}}{2^{d+1}\pi^{\frac{d}{2}}T(|A|b^2)^{-s}\Gamma(s)}\int_0^\infty i^{1-\frac{d}{2}}\tau_b^{s-\frac{d+2}{2}}e^{\frac{C}{b^2}\tau_b}\right.$$
$$\left.\int [g_{L_1,b}(\tau_b, x_b, x_b) - g_{L_0,b}(\tau_b, x_b, x_b)]\,dx_b d\tau_b\right\}. \tag{5.1.11}$$

Interestingly, in all cases considered in this work C is proportional to b^2, so we can rightfully substitute

$$C = C_b b^2 \tag{5.1.12}$$

and obtain

$$\Delta E = \Re\left\{\lim_{s\to 0}\frac{\partial}{\partial s}\frac{\hbar c b^d l^{d-1}}{2^{d+1}\pi^{\frac{d}{2}}T(|A|b^2)^{-s}\Gamma(s)}\int_0^\infty i^{1-\frac{d}{2}}\tau_b^{s-\frac{d+2}{2}}e^{C_b\tau_b}\right.$$
$$\left.\int [g_{L_1,b}(\tau_b, x_b, x_b) - g_{L_0,b}(\tau_b, x_b, x_b)]\,dx_b d\tau_b\right\}. \tag{5.1.13}$$

Unfortunately, extracting C_b out of the Mellin transform is not a trivial task. Nevertheless at this point we have most of physical parameters of the classical system extracted out of the Green function. Apart from qualitative estimations of quantum corrections, it helps us in choosing the renormalization parameter r^2, since the logarithmic divergences will arise from differentiation of exponential components of the zeta-function. This does not concern divergence in T only - it is also reasonable to assume, that if the classical solution vanishes, corrections should vanish as well. This means, that r^2 should contain b^2 The same argument can be used to cut divergences in all physically relevant parameters, thus it seems valid to propose for the

renormalization factor to cancel the whole $(|A|b^2)^{-s}$ term. Yet, there is no strict way of choosing the value of r^2, so in practice we will fit its value to recover well known results in the cases obtainable by different methods.

At this point it is also worth noting, that in our chosen dimensionless variables c is linearly dependent on T, so it cancels out the T component in the denominator. This would not be the case, if we didn't include the kinetic energy component in the action integral and later on in calculation of energy corrections. Combined with the strong dependence of the result on the number of spatial dimensions included it indicates that one should never omit any elements in the action integral even if it would be valid for the classical solution.

One of the most important findings going beyond the chosen class of potentials, if we consider a $\tau_A = |A|\tau$ scaling, is that quantum corrections to energy do not depend on the scale of the classical system. If we were to amplify the action integral by a constant factor, it would have no impact on the quantum corrections. This quality will also be present in other semiclassical methods such as harmonic oscillator expansion of the potential (method used for example in [16] and explained in detail later in [74]). The results from this subsection were published in [110].

5.2 Single kink case

5.2.1 General results

Let us first construct potentials for single kink solutions of Sine-Gordon and ϕ^4 models in dimensionless form of (3.3.44) with respective m coefficients calculated separately for specific physical models (see Chapter 2):

$$\frac{a^2 V_{SG}''(\theta)}{G} = -m_\theta^2 \cos(\theta), \tag{5.2.14}$$

which for a static kink gives

$$\frac{a^2 V_{SG}''(\theta)}{G} = -m_\theta^2[1 - 2\mathrm{sech}^2(m_\theta x)], \tag{5.2.15}$$

and

$$\frac{a^2 V_{\phi^4}''(\phi)}{G} = m_\phi^2 - 6\frac{m_\phi^2}{V^2}\phi^2, \tag{5.2.16}$$

which for a static kink gives

$$\frac{a^2 V_{\phi^4}''(\phi)}{G} = -\frac{m_\phi^2}{2}\left[4 - 6\mathrm{sech}\left(\frac{m_\phi}{\sqrt{2}}x\right)\right]. \tag{5.2.17}$$

As can be seen, they both fit into the form

$$u(x) = U_1 sech^2(bx) + U_0 \tag{5.2.18}$$

with $b = m_\theta$ or $b = \frac{m_\phi}{\sqrt{2}}$ respectively. Using the algorithm described in Section 4.4 we will obtain green function diagonal for specific values of U_1. Since we will be able to obtain exact results for those cases, we won't need to rescale the x variable and the Green function diagonal by b. We will however use the proposed normalising parameter r^2 value. We proceed with substitution

$$z = sech^2(bx), \tag{5.2.19}$$

$$(z'_x)^2 = 4b^2 z^2 (1 - z). \tag{5.2.20}$$

Green function diagonal equation takes form (before the inclusion of $-i$ factor)

$$2G[G''_{zz}(z'_x)^2 + G'_z z''_{xx}] - (G'_z z'_x)^2 - 4(U_1 z + U_0 - p)G^2 + 1 = 0. \tag{5.2.21}$$

We seek solutions of form (4.2.20). Existence and complexity of solutions depends on the value of U_1. An analog of equation (4.3.43) will take form

$$-8b^2 M(M - 1) - 12b^2 M + 4b^2 M^2 - 4U_1 = 0, \tag{5.2.22}$$

$$U_1 = -b^2 M^2 - b^2 M. \tag{5.2.23}$$

$M = 0$ gives $U_1 = 0$ and thus represents vacuum potential

$$G_{M=0} = \frac{1}{2\sqrt{U_0 - p}}. \tag{5.2.24}$$

$M = 1$ gives $U_1 = -2b^2$ and is related to Sine-Gordon soliton ($U_0 = b^2$ and $b = m_\theta$ for easy plane ferromagnet)

$$G_{M=1} = \frac{p - b^2 z}{2p\sqrt{b^2 - p}}. \tag{5.2.25}$$

$M = 2$ gives $U_1 = -6b^2$ and is related to ϕ^4 kink ($U_0 = 4b^2$ and $b = \frac{m_\phi}{\sqrt{2}}$ for easy axis ferromagnet)

$$G_{M=2} = \frac{p^2 - 3b^2 p(1 + z) + 9b^4 z^2}{2p(p - 3b^2)\sqrt{4b^2 - p}}. \tag{5.2.26}$$

Solutions of an arbitrary order can be obtained as well.

We proceed with the regularization scheme 3.3 with vacuum potential chosen so that the square roots in the denominator coincide. At this point we also include the results of Section 4.5.1

$$\gamma_{M=j} = \mathcal{L}^{-1}\left[\int_{-\infty}^{\infty} i\{G_{M=j}(ip, x) - G_{M=0}(ip, x)\}\, dx\right] \tag{5.2.27}$$

with $\mathcal{L}^{-1}$ representing inverse Laplace transform.

$$\gamma_{M=1}(\tau_A) = -Erf(b\sqrt{i\tau_A}), \tag{5.2.28}$$

$$\gamma_{M=2}(\tau_A) = -e^{3b^2 i\tau_A} Erf(b\sqrt{i\tau_A}) - Erf(2b\sqrt{i\tau_A}). \tag{5.2.29}$$

After accounting for the remaining variables we calculate zeta-function. All calculations assume $\Re(s) < \frac{d}{2}$

$$\zeta_{M=j}(s) = \frac{1}{|A|^s \Gamma(s)} \int_0^\infty \tau_A^{s-1} \gamma_2(\tau_A) \gamma_3(\tau_A) \gamma_{M=j}(\tau_A) d\tau_A, \tag{5.2.30}$$

which for particular values of M gives

$$\zeta_{M=1}(s) = -\frac{icl^{d-1}b^d \Gamma\left(\frac{1-d}{2} + s\right)}{2^{d-1} \pi^{\frac{1+d}{2}} (-Ab^2)^s (d - 2s) \Gamma(s)} \tag{5.2.31}$$

and

$$\zeta_{M=2} = -\frac{icb^d l^{d-1} \Gamma\left(\frac{1-d}{2} + s\right)}{2^{d-1} (-Ab^2)^s \pi^{\frac{d+1}{2}} \Gamma(s)}$$
$$\left[3^{-s+\frac{d-1}{2}} \, {}_2F_1\left(\frac{1}{2}, s - \frac{d-1}{2}, \frac{3}{2}, -\frac{1}{3}\right) + 2^{-2s+d} \frac{1}{d - 2s} \right], \tag{5.2.32}$$

where $_2F_1$ is the hypergeometric function. Derivatives are then calculated separately for each value of d. At this point it is important to stress, that Γ function is singular for all non-positive integers. For $d = 1$ and $d = 3$ it is not a problem, since we can use reduction formulae for division of Euler's Gamma functions. Regularization factor r for those cases was chosen to cut all logarithmic divergences (namely $\ln(-Ab^2)$). On the other hand for $d = 2$ the $\Gamma(s)$ in the denominator is the only singular part of ζ function and results in $\lim_{s \to 0} \zeta(s) = 0$. Yet, the derivative is non-zero and finite. The simplest way to obtain the result is to multiply ζ function by $\frac{s}{s}$ (shown below on $\zeta_{M=1}$)

$$\zeta_{M=1}(s) = -\frac{iclb^2 s \Gamma\left(s - \frac{1}{2}\right)}{2\pi^{\frac{3}{2}} (-Ab^2)^s (2 - 2s) \Gamma(s+1)}. \tag{5.2.33}$$

From this form it is very easy to calculate the derivative, which at $s = 0$ has only one non-zero component

$$\frac{\partial \zeta_{M=1}}{\partial s}(0) = -\frac{iclb^2 \Gamma\left(-\frac{1}{2}\right)}{4\pi^{\frac{3}{2}}}. \tag{5.2.34}$$

The most important feature of this case (as well as for any other value of M) is lack of the logarithmic component, which means that the results are independent of the regularization factor r. Quantum corrections to energy will finally take form given in Table 5.1. It is important to stress, that since c in chosen set of dimensionless variables is linearly dependent on T, corrections will not depend on T after final substitution of all parameters as will be seen in next subsections. Due to the uncertainty of the sign of corrections resulting from Green function diagonal equation there might be a disagreement by a factor of -1 with other methods, nevertheless the sign is consequent for all values of d.

The corrections for Sine-Gordon soliton are in agreement with works of Konoplich [72] having in mind, that Konoplich assumed most physical constants to be

Table 5.1: Energy corrections for kink solutions.

d	Sine-Gordon	ϕ^4
1	$-\frac{\hbar bc}{T\pi}$	$\frac{\hbar bc}{2T\pi}\left(6\ln(2)+\frac{\pi}{\sqrt{3}}-6\right)$
2	$\frac{\hbar b^2 cl}{4T\pi}$	$\frac{\hbar b^2 cl}{2T\pi}\left(3+\frac{3}{2}\arcsin\left(\frac{1}{\sqrt{3}}\right)\right)$
3	$\frac{5\hbar b^3 cl^2}{72T\pi^2}$	$-\frac{\hbar b^3 cl^2}{12T\pi^2}\left(18\ln(2)-18+\frac{\pi}{\sqrt{3}}\right)$

equal to 1 and used imaginary time formalism, so the two-dimensional case in his work corresponds to the $1+1$ dimensional case in our work. Konoplich also left the mass scale (r^2 in our work) undetermined. The $1+1$ dimensions case is also identical to the one obtained by Faddeev [111], where the mass scale is not explicitly used, but the action integral is implicitly regularized in a similar way.

In the case of ϕ^4 kink, we also obtained the same results as Konoplich (with the same considerations as for Sine-Gordon case), but our intuitive choice of normalising factor r^2 does not agree with works of Dashen et al. [16] and later publications confirming that result (as for example [112, 80]). A different choice for r^2 would allow us to recover proper corrections to energy (namely one leading to $-4Ab^2 = 1$), yet within the zeta-function regularization scheme there is so far no direct method of finding the correct normalising factor. Nevertheless, if we fit the $1+1$ case with works of Dashen et al., we unambiguously expand the results to $1+2$ and $1+3$ cases and have a solid basis for quantization of periodic and quasi-periodic fields of Sine-Gordon and ϕ^4 systems as shown in Section 5.3. Therefore following the works of Dashen et al. we obtain for the ϕ^4 proper results (see Table 5.2) It is of note,

Table 5.2: Energy corrections for ϕ^4 kinks with proper regularization.

d	ΔE
1	$\frac{\hbar bc}{2T\pi}\left(\frac{\pi}{\sqrt{3}}-6\right)$
2	$\frac{\hbar b^2 cl}{2T\pi}\left(3+\frac{3}{2}\arcsin\left(\frac{1}{\sqrt{3}}\right)\right)$
3	$\frac{\hbar b^3 cl^2}{4T\pi^2}\left(6-\frac{\pi}{\sqrt{3}}\right)$

that the result for $3+1$ dimensions is in agreement with the publication by Ventura [71], which proves that our choice of r^2 is correct.

5.2.2 Domain walls

The results can be directly applied to magnetic domain walls in ferromagnets with axial anisotropy. As was shown in section 2, ferromagnets with easy axis anisotropy can be modelled by a wave equation with ϕ^4 potential and those with easy plane anisotropy by Sine-Gordon equation. In both cases domain walls are expressed as single kink solutions also shown in Section 2. With the classical energy E_c treated as

Table 5.3: Energy corrections for single domain walls.

	Easy plane	Easy axis
classic solution	$2\arcsin\left(\operatorname{tgh}\left(m_\theta x\right)\right)-\pi$	$m_\phi\sqrt{\frac{-6J}{8D+g\mu_B B}}\operatorname{tgh}\left(\frac{m_\phi}{\sqrt{2}}x\right)$
m	$\sqrt{\frac{g\mu_B B}{J}}$	$\sqrt{-\frac{2D+g\mu_B B}{J}}$
c	$\sqrt{\frac{2JDT^2}{\hbar^2}}$	$\sqrt{\frac{Jg\mu_B B T^2}{\hbar^2}}$
E_c	$8\sqrt{Jg\mu_B B}$	$\frac{4\sqrt{-2J}(2D+g\mu_B B)^{\frac{3}{2}}}{8D+g\mu_B B}$
ΔE_1	$-\sqrt{\frac{2Dg\mu_B B}{\pi^2}}$	$\frac{1}{2\sqrt{2}\pi}\sqrt{-(2D+g\mu_B B)g\mu_B B}\left(\frac{\pi}{\sqrt{3}}-6\right)$
ΔE_2	$l\sqrt{\frac{Dg^2\mu_B^2 B^2}{8J\pi^2}}$	$-l\frac{2D+g\mu_B B}{4\pi}\sqrt{\frac{g\mu_B B}{J}}\left(3+\frac{3}{2}\arcsin\left(\frac{1}{\sqrt{3}}\right)\right)$
ΔE_3	$\frac{5}{36\pi^2}l^2\sqrt{\frac{Dg^3\mu_B^3 B^3}{2J^2}}$	$l^2\frac{\sqrt{-(2D+g\mu_B B)^3 g\mu_B B}}{8\sqrt{2}J\pi^2}\left(6-\frac{\pi}{\sqrt{3}}\right)$

per single chain of atoms. The above results were published in [113, 114]. The most important conclusion is, that on quantum level the energy is highly dependent on the number of spatial dimensions taken into account, which is not seen in the classical model. This means, that magnetic domain walls in bulk material (three spatial dimensions) would interact differently with an external magnetic field (figure 5.3) than for example those in thin ferromagnetic films (figure 5.2). This should be most visible for easy axis ferromagnetic domain walls with B very near the $-\frac{2D}{g\mu_B}$ limit, where for planar and linear ferromagnets the quantum corrections would dominate over the classical energy. Moreover, for linear easy axis ferromagnets overall energy of a domain wall would be negative in certain magnetic field range (figure 5.1 with blue line representing the classical energy and red line representing corrected energy in arbitrary units as a function of magnetic field scaled so that $-\frac{2D}{g\mu_B}=1$ and with $D=-0.3J$), which would imply spontaneous creation of domain walls. As such it is insufficient to consider a singular domain wall and interaction between them has to be accounted for. This phenomenon will be further investigated in Section 5.3. In the case of easy plane ferromagnetic domain walls in $d=1$ case corrections to classic energy ratio gives

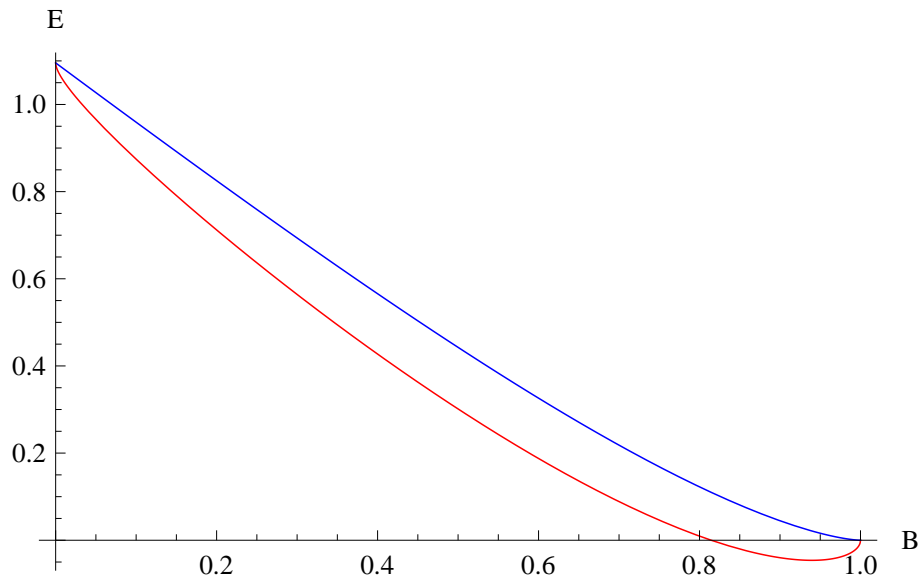$$\frac{\Delta E_1}{E_c}=\frac{1}{8\pi}\sqrt{\frac{2D}{J}}. \tag{5.2.35}$$

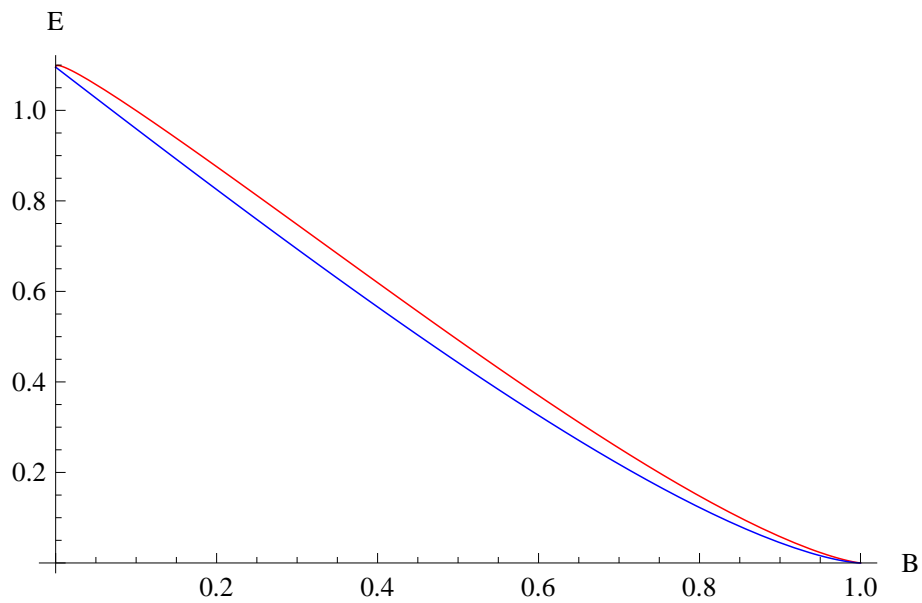Figure 5.1: Easy axis domain wall energy for $d = 1$ in arbitrary units.

Figure 5.2: Easy axis domain wall energy for $d = 2$ in arbitrary units.

For $(CH_3)_4NNiBr_3$ (J and D parameters according to [115]) this would give a ratio of 0.0346, which might be noticed in experiments. Corrections for planar and bulk ferromagnets will be less pronounced due to the $D >> g\mu_B B$ requirement of the easy plane approximation, which usually implies $J >> g\mu_B B$ at least in materials for which the continuum approximation is valid.
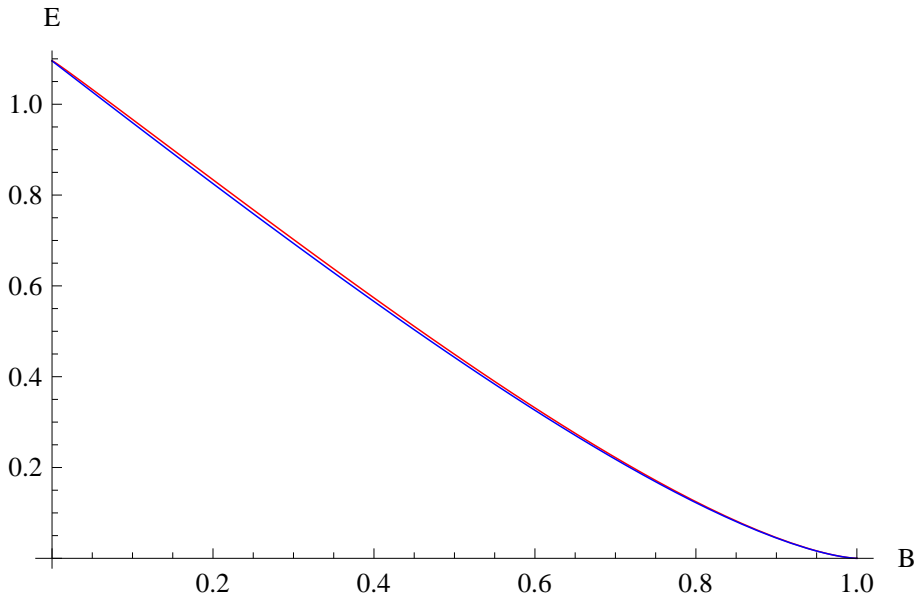
Figure 5.3: Easy axis domain wall energy for $d = 3$ in arbitrary units.

5.3 Cnoidal wave case

5.3.1 General results

The results of this and next subsection were published in [110]. We will start with calculating potentials for Sine-Gordon and ϕ^4 cnoidal solutions

$$\frac{a^2 V''_{SG}(\theta)}{G} = m_\theta^2 \cos \theta, \tag{5.3.36}$$

which gives

$$\frac{a^2 V''_{SG}(\theta)}{G} = m_\theta^2 \left[2k^2 - 1 - 2k^2 \text{cn}^2(m_\theta x; k) \right], \tag{5.3.37}$$

and

$$\frac{a^2 V''_{\phi^4}(\phi)}{G} = -m_\phi^2 + 6 \frac{m_\phi^2}{V^2} \phi^2, \tag{5.3.38}$$

which gives

$$\frac{a^2 V''_{\phi^4}(\phi)}{G} = \frac{m_\phi^2}{1 + k^2} \left[5k^2 - 1 - 6k^2 \text{cn}^2 \left(\frac{m_\phi}{\sqrt{1 + k^2}} x; k \right) \right]. \tag{5.3.39}$$

Both potentials can be written in a form

$$U(x) = U_1 \text{cn}^2(bx; k) + U_0, \tag{5.3.40}$$

where cn is a Jacobi elliptic function, $b = m_\theta$ for the Sine-Gordon case and $b = \frac{m_\phi}{\sqrt{1+k^2}}$ for the ϕ^4 case. We don't have explicit solutions for energy corrections, thus we will use the scaling presented in Section 5.1. As in a single soliton case we use algorithm

4.4 with substitution

$$z = \text{cn}^2(x_b; k), \qquad (5.3.41)$$
$$(z'_{x_b})^2 = 4z(1 - z)(k^2 z + 1 - k^2). \qquad (5.3.42)$$

As before we obtain a relation between U_1 and order of the polynomial P

$$U_1 = -k^2(M^2 + M). \qquad (5.3.43)$$

It is important to note, that $k = 1$ gives back single soliton case and $M = 0$ gives an identical vacuum solution for all values of k. $M = 1$ gives $U_1 = -2k^2$ and is related to Sine-Gordon cnoidal wave ($U_0 = 2k^2 - 1$)

$$G_{M=1} = \frac{p - k^2 z}{2\sqrt{p[p - (k^2 - 1)][k^2 - p]}}. \qquad (5.3.44)$$

$M = 2$ gives $U_1 = -6k^2$ and is related to three different ϕ^4 cnoidal waves ($U_0 = (5k^2 - 1)$)

$$G_{M=2} = \frac{p^2 + 9k^2[1 + k^2(-1 + z)]z - 3[p + k^2 pz]}{2\sqrt{[3 - p]p[p - 3k^2][(p - 1 - k^2)^2 - 4(1 - k^2 + k^4)]}}. \qquad (5.3.45)$$

In each case we pick the vacuum potential so it would coincide with the highest root of the denominator of the Green function diagonal ($U_0 = k^2$ for Sine-Gordon model and $U_0 = 1 + k^2 + 2\sqrt{1 - k^2 + k^4}$ for ϕ^4), which is also in both cases the only root, which does not converge with any other at $k \to 1$ limit.

The key problem is the inverse Laplace transform of the Green function diagonal, since $G_{M>0}$ generates complex space of non-zero genus. There is however a definition of bilateral inverse Laplace transform for which it is sufficient for the Green function diagonal to be analytic in a $o_l < \Re(p) < o_p$ area for a given pair of real constants o_l and o_p

$$g(x_b, x_b, \tau_b) = \frac{1}{2\pi i} \int_{o-i\infty}^{o+i\infty} e^{p\tau_b} iG_{M=j}(ip, x_b) dp, \qquad (5.3.46)$$

where $o_l < o < o_p$ and scaling by i is included. Disregarding the choice of the sign of the square root we have only two choices for o (either $o < 0$ or $o > 0$), since all the singularities will lie on the imaginary axis. Considering the condition $\forall_{\tau<0} g(x, x_0, \tau) = 0$ we have to choose $o > 0$. There still remains the problem of evaluating the integral. We can however expand the G function around the $k = 1$ case or more precisely expand only the problematic part of said function omitting the elliptic integrals emergent from the integration over the spatial variable x. We will now showcase the procedure using the Sine-Gordon quasi-periodic ($\varphi(x_b + 2\mathcal{K}(k)) = \varphi(x_b) + 2\pi$) solution. Let us first subtract the vacuum solution from the Green function diagonal integrate over the period of the potential ($2\mathcal{K}(k)$) and substitute $iG(ip, x)$ as the Green function diagonal

$$\hat{\gamma}_{M=1}(p) = \int_{-\mathcal{K}(k)}^{\mathcal{K}(k)} (iG_{M=1}(ip, x_b) - iG_{M=0}(ip, x_b)) \, dx_b, \qquad (5.3.47)$$

which gives

$$\hat{\gamma}_{M=1}(p) = i\frac{-\mathcal{E}(k) + \left(-1 + k^2 + ip - ip\sqrt{\frac{1-k^2}{ip}+1}\right)\mathcal{K}(k)}{\sqrt{(ip+1-k^2)(k^2-ip)ip}}. \tag{5.3.48}$$

We now expand $\hat{\gamma}$ function in a power series with respect to k at $k = 1$ treating both elliptic integrals as independent parameters. We obtain

$$\begin{aligned}
\hat{\gamma}_{M=1}(p) &\approx -i\frac{\mathcal{E}(k)}{ip\sqrt{1-ip}} + i(k-1)\frac{\mathcal{E}(k)(2ip-1) + 3\mathcal{K}(k)(p^2+ip)}{(-1+ip)p^2\sqrt{1-ip}} \\
&\quad -i(k-1)^2\left[\frac{\mathcal{E}(k)(-3+7ip+5p^2+2ip^3)}{(-1+ip)^2p^3\sqrt{ip-1}}\right.\\
&\quad \left.+\frac{\mathcal{K}(k)(7ip+17p^2-7ip^3+3p^4)}{(-1+ip)^2p^3\sqrt{ip-1}}\right] + \dots \; .
\end{aligned} \tag{5.3.49}$$

Inverse Laplace transform of this function is easily calculated, since any function of form

$$\frac{\Pi_{j=1}^N(p-a_j)}{\Pi_{j=1}^N(p-b_j)\sqrt{b_0-p}} \tag{5.3.50}$$

can be rewritten as a sum of

$$\frac{1}{(p-b_j)^n\sqrt{b_0-p}} \tag{5.3.51}$$

for which inverse Laplace transforms are well known. We will obtain

$$\begin{aligned}
\gamma_{M=1}(\tau_b) &= -\mathcal{E}(k)\text{Erf}(\sqrt{i\tau_b}) \\
&\quad +(k-1)\left\{-\frac{1}{\sqrt{\pi}}e^{-i\tau_b}\mathcal{E}(k)\sqrt{i\tau_b} + \frac{1}{2}[6\mathcal{K}(k) + \mathcal{E}(k)(1+2i\tau_b)]\text{Erf}(\sqrt{i\tau_b})\right\} \\
&\quad -\frac{1}{8}(k-1)^2\left\{\frac{2}{\sqrt{\pi}}e^{-i\tau_b}\sqrt{i\tau_b}[8\mathcal{K}(k)(1+i\tau_b) + 5\mathcal{E}(k)(1+2i\tau_b)]\right.\\
&\quad \left.+\text{Erf}[\sqrt{i\tau_b}]\left[4\mathcal{K}(k)(1-14i\tau_b) + \mathcal{E}(k)(-5+4i\tau_b+12\tau_b^2)\right]\right\} \\
&\quad + \dots \; .
\end{aligned} \tag{5.3.52}$$

All elements present in this function (as well as in all further elements of the Taylor series) can be in general divided into two groups (with n as a non-negative integer):

1. $\text{Erf}(\sqrt{i\tau_b})\,(i\tau_b)^n$,

2. $(i\tau_b)^{n+\frac{1}{2}}\,e^{-i\tau_b}$.

Since we still need to multiply $\gamma_{M=1}$ by γ_{L_2} (5.1.9) and γ_{L_3} (5.1.10) we will be performing Mellin transform on elements:

1. $i\text{Erf}(\sqrt{i\tau_b})\,(i\tau_b)^{n-\frac{d}{2}}$,

2. $i\,(i\tau_b)^{n+\frac{1-d}{2}}\,e^{-i\tau_b}$.

To construct the final form of the zeta-function we need to find Mellin transforms of those two types of elements. In essence we need to solve integrals:

1. $\frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Erf}(\sqrt{i\tau_b}) i^{n+1-\frac{d}{2}} \tau_b^{s-\frac{d+2}{2}+n} d\tau_b$,

2. $\frac{1}{\Gamma(s)} \int_0^\infty i^{n+\frac{3}{2}-\frac{d}{2}} \tau_b^{s-\frac{d+2}{2}+n+\frac{1}{2}} e^{-i\tau_b} d\tau_b$.

It is evident, that there are possible convergence issues at either of the limits. Yet, those integrals can still be evaluated as a limit of slightly different integrals:

1. $\frac{1}{\Gamma(s)} \lim_{\epsilon \to 0_+} \int_0^\infty \mathrm{Erf}(\sqrt{i\tau_b}) i^{n+1-\frac{d}{2}} \tau_b^{s-\frac{d+2}{2}+n} e^{-\epsilon\tau_b} d\tau_b$,

2. $\frac{1}{\Gamma(s)} \lim_{\epsilon \to 0_+} \int_0^\infty i^{n+\frac{3}{2}-\frac{d}{2}} \tau_b^{s-\frac{d+2}{2}+n+\frac{1}{2}} e^{-i\tau_b} e^{-\epsilon\tau_b} d\tau_b$

as long as τ_b exponent is higher than -1. This will most definitely not be the case, when we will try to obtain the limit of the derivative of zeta-function at $s \to 0$. Still, if the resulting function is well defined even outside of this condition we can use it as an analytical continuation. As a result we will obtain

1. $\frac{2i^{1-s}\Gamma(\frac{1-d}{2}+n+s)}{\sqrt{\pi}(d-2(n+s))\Gamma(s)}$,

2. $\frac{i^{1-s}\Gamma(\frac{1-d}{2}+n+s)}{\Gamma(s)}$.

With this result we can finally construct the zeta-function

$$
\begin{aligned}
\zeta(s) \;=\; & \frac{icb^d l^{d-1}}{2^d \pi^{\frac{d+1}{2}} (-Ab^2)^{-s} \Gamma(s)} \left\{ -\mathcal{E}(k) \frac{2\Gamma(\frac{1-d}{2}+s)}{d-2s} \right. \\
& + (k-1) \left[-\mathcal{E}(k)\Gamma(\frac{1-d}{2}+s) + 6\mathcal{K}(k)\frac{\Gamma(\frac{1-d}{2}+s)}{d-2s} \right. \\
& \left. + \mathcal{E}(k) \left(\frac{\Gamma(\frac{1-d}{2}+s)}{d-2s} + 2\frac{\Gamma(\frac{3-d}{2}+s)}{d-2(1+s)} \right) \right] \\
& - \frac{(k-1)^2}{8} \left[16\mathcal{K}(k) \left(\Gamma(\frac{1-d}{2}+s) + \Gamma(\frac{3-d}{2}+s) \right) \right. \\
& + 10\mathcal{E}(k) \left(\Gamma(\frac{1-d}{2}+s) + 2\Gamma(\frac{3-d}{2}+s) \right) \\
& + 4\mathcal{K}(k) \left(\frac{\Gamma(\frac{1-d}{2}+s)}{d-2s} - 14\frac{\Gamma(\frac{3-d}{2}+s)}{d-2(1+s)} \right) \\
& \left.\left. + \mathcal{E}(k) \left(\frac{-5\Gamma(\frac{1-d}{2}+s)}{d-2s} + 4\frac{\Gamma(\frac{3-d}{2}+s)}{d-2(1+s)} - 12\frac{\Gamma(\frac{5-d}{2}+s)}{d-2(2+s)} \right) \right] \right\} \\
& + \dots
\end{aligned}
\tag{5.3.53}
$$

When calculating the derivative of the ζ function we will be concerned with two kinds of elements:

1. $\frac{\Gamma(\frac{1-d}{2}+n+s)}{(d-2(n+s))\Gamma(s)}$,

2. $\frac{\Gamma(\frac{1-d}{2}+n+s)}{\Gamma(s)}$,

since our choice of regularization factor r ensures $-Ab^2 = 1$. One has to be careful, because Euler's Gamma function is singular at 0 as well as all negative integers. This means that the derivatives have to be calculated separately for specific values of d just as in the single soliton case

$$
\begin{aligned}
\Delta E_1(k) \;=\; & \frac{\hbar c b}{4T\pi}\Big\{-4\mathcal{E}(k) + 12(k-1)\mathcal{K}(k) \\
& -\frac{(k-1)^2}{8}\left[80\mathcal{K}(k) - 10\mathcal{E}(k)\right]\Big\} + \ldots,
\end{aligned}
\tag{5.3.54}
$$

$$
\begin{aligned}
\Delta E_2(k) \;=\; & \frac{\hbar c b^2 l}{8T\pi}\Big\{2\mathcal{E}(k) + (k-1)\left[-6\mathcal{K}(k) + \mathcal{E}(k)(-\ln(4))\right] \\
& -\frac{(k-1)^2}{8}\left[-4\mathcal{K}(k)(5 + 14\ln(2)) + 4\mathcal{E}(k)(2 + \ln(2))\right]\Big\} + \ldots,
\end{aligned}
\tag{5.3.55}
$$

$$
\begin{aligned}
\Delta E_3(k) \;=\; & \frac{\hbar c b^3 l^2}{16T\pi^2}\Big\{\frac{10}{9}\mathcal{E}(k) + (k-1)\left[-\frac{15}{3}\mathcal{K}(k) + \mathcal{E}(k)\left(-5 + \frac{5}{9}\right)\right] \\
& -\frac{(k-1)^2}{8}\left[\mathcal{K}(k)\left(-\frac{20}{9} - 128\right) + \mathcal{E}(k)\left(10 + \frac{25}{9}\right)\right]\Big\} + \ldots.
\end{aligned}
\tag{5.3.56}
$$

As can be seen, the results converge to single soliton case for $k \to 1$ as should be expected. Arbitrarily many elements of the Taylor series can be evaluated using the same method and thus an arbitrary level of precision can be obtained. As yet, there are no clear results on the convergence radius of the series, which also makes approximation inaccuracies difficult to assess. Combined with the difficulty of recreating numerical calculations of Pawellek [79] it regrettably makes comparison of results difficult. It is also very important to notice, that because of elements containing $\frac{\Gamma\left[\frac{1}{2}+s\right]}{2s\Gamma[s]}$ the results for $d = 2$ depend on the value of r^2 unlike the $k \to 1$ limit.

Using the same procedures we obtained energy corrections for the cnoidal solution of the ϕ^4 model (with regularization factor chosen as in the single kink case)

$$
\begin{aligned}
\Delta E_1(k) \;=\; & -\frac{\hbar bc}{4T\pi}\Big\{2\mathcal{E}(k)(6 - \frac{\pi}{\sqrt{3}}) + (k-1)\left[-\frac{2\pi}{\sqrt{3}}\mathcal{K}(k) - 6(\mathcal{E}(k) - 2\mathcal{K}(k))\right] \\
& +(k-1)^2\left[-15\mathcal{K}(k) + \frac{3}{2}\mathcal{E}(k) + \frac{\pi}{\sqrt{3}}(\mathcal{K}(k) - 2\mathcal{E}(k))\right]\Big\} \\
& + \ldots,
\end{aligned}
\tag{5.3.57}
$$

$$\Delta E_2(k) = -\frac{\hbar l c b^2}{8T\pi} \Big\{ -\mathcal{E}(k)(12 + 3\ln(3)) - (k-1)\frac{3}{2}(\mathcal{E}(k) + 2\mathcal{K}(k))(4 + \ln(3))$$
$$+ (k-1)^2 \frac{3}{8} \left[\mathcal{E}(k)(6 - 33\mathrm{arccoth}(2)) + \mathcal{K}(k)(-40 + 6\ln(3)) \right] \Big\}$$
$$+ \dots, \tag{5.3.58}$$

$$\Delta E_3(k) = -\frac{\hbar l^2 b^3 c}{48\pi^2 T} \Big\{ 4\mathcal{E}(k)(\sqrt{3}\pi - 18) +$$
$$4(k-1)\left[\sqrt{3}(\mathcal{K}(k) + \mathcal{E}(k))\pi - 9(2\mathcal{K}(k) + \mathcal{E}(k)) \right] -$$
$$(k-1)^2 \left[2\mathcal{K}(k)(7 + 2\sqrt{3}\pi) - \mathcal{E}(k)(14\sqrt{3}\pi - 27) \right] \Big\} + \dots. \tag{5.3.59}$$

One has to keep in mind that $b = \frac{m_\phi}{\sqrt{1+k^2}}$. As before corrections for $d = 2$ become dependent on the value of the normalising factor r^2. We also obtain proper single kink results in the $k \to 1$ limit.

5.3.2   Stripe domains

In the context of easy axis and easy plane magnetic chains cnoidal solutions of Sine-Gordon and ϕ^4 models represent a cross-section of a stripe domain structure. Width of the particular domains is equal to $2\mathcal{K}(k)$ in units of x_b or $\frac{2}{b}\mathcal{K}(k)$ in the units of x (dimensionless units normalized to the crystal lattice constant in the direction of the chain). We substitute model parameters in the same way as in Section 5.2.2 and obtain

$$\Delta E_1(k) = \sqrt{\frac{Dg\mu_B B}{8\pi^2}} \Big\{ -4\mathcal{E}(k) + 12(k-1)\mathcal{K}(k)$$
$$-\frac{(k-1)^2}{8} \left[80\mathcal{K}(k) - 10\mathcal{E}(k) \right] \Big\} + \dots, \tag{5.3.60}$$

$$\Delta E_2(k) = \frac{l}{4\pi} \sqrt{\frac{Dg^2 \mu_B^2 B^2}{2J}} \Big\{ 2\mathcal{E}(k) + (k-1)\left(-6\mathcal{K}(k) + \mathcal{E}(k)(-\ln(4)) \right)$$
$$-\frac{(k-1)^2}{2} \left[-\mathcal{K}(k)(5 + 7\ln(4)) + \mathcal{E}(k)(2 + \ln(2)) \right] \Big\} + \dots, \tag{5.3.61}$$

$$\Delta E_3(k) = \frac{l^2}{8\pi^2} \sqrt{\frac{Dg^3 \mu_B^3 B^3}{2J^2}} \Big\{ \frac{10}{9}\mathcal{E}(k) + (k-1)\left[-\frac{15}{3}\mathcal{K}(k) + \mathcal{E}(k)\left(-5 + \frac{5}{9} \right) \right]$$
$$-\frac{(k-1)^2}{8} \left[\mathcal{K}(k)\left(-\frac{20}{9} - 128 \right) + \mathcal{E}(k)\left(10 + \frac{25}{9} \right) \right] \Big\} + \dots \tag{5.3.62}$$

for the easy plane domain walls and

$$
\begin{aligned}
\Delta E_1(k) &= -\frac{\sqrt{-(2D + g\mu_B B)g\mu_B B}}{4\sqrt{k^2 + 1}\pi} \left\{ 2\mathcal{E}(k)[6 - \frac{\pi}{\sqrt{3}}] \right. \\
&\quad + (k-1)\left[-\frac{2\pi}{\sqrt{3}}\mathcal{K}(k) - 6(\mathcal{E}(k) - 2\mathcal{K}(k)) \right] \\
&\quad \left. + (k-1)^2 \left[-15\mathcal{K}(k) + \frac{3}{2}\mathcal{E}(k) + \frac{\pi}{\sqrt{3}}(\mathcal{K}(k) - 2\mathcal{E}(k)) \right] \right\} + \dots,
\end{aligned}
$$
(5.3.63)

$$
\begin{aligned}
\Delta E_2(k) &= l\frac{2D + g\mu_B B}{8\pi(k^2 + 1)}\sqrt{\frac{g\mu_B B}{J}} \left\{ -\mathcal{E}(k)[12 + 3\ln(3)] \right. \\
&\quad - (k-1)\frac{3}{2}[\mathcal{E}(k) + 2\mathcal{K}(k)][4 + \ln(3)] \\
&\quad \left. + (k-1)^2\frac{3}{8}[\mathcal{E}(k)(6 - 33\text{arccoth}(2)) + \mathcal{K}(k)(-40 + 6\ln(3))] \right\} + \dots,
\end{aligned}
$$
(5.3.64)

$$
\begin{aligned}
\Delta E_3(k) &= -l^2\frac{\sqrt{-(2D + g\mu_B B)^3 g\mu_B B}}{48J\pi^2(k^2 + 1)^{\frac{3}{2}}} \left\{ 4\mathcal{E}(k)[\sqrt{3}\pi - 18] \right. \\
&\quad + 4(k-1)[\sqrt{3}(\mathcal{K}(k) + \mathcal{E}(k))\pi - 9(2\mathcal{K}(k) + \mathcal{E}(k))] \\
&\quad \left. - (k-1)^2[2\mathcal{K}(k)(7 + 2\sqrt{3}\pi) - \mathcal{E}(k)(14\sqrt{3}\pi - 27)] \right\} + \dots
\end{aligned}
$$
(5.3.65)

for easy axis domain walls.

The most important aspect of the results is that the dependence on material parameters and external magnetic field is exactly the same as in the singular domain wall case and is not affected by our approximations regarding the Green function diagonal as a result of our rescaling methods shown in Section 5.1. Our power series expansion does affect the dependence on the elliptic modulus. Until we find the convergence radius and consequently estimate the inaccuracies arising due to those approximations, the results can at best be considered as qualitative estimates of width dependence of stripe domain energy. If we were to solve this problem, we would be able to determine interaction energy of domain walls.

Quantum corrections for easy plane domain walls are significantly smaller than the classical energy even for comparatively large anisotropy ($\frac{D}{J}$) considering the easy plane approximation constrain $D >> g\mu_B B$. Therefore we can conclude, that in this case classical model gives accurate qualitative predictions.

As for the easy axis case, we already established, that near the critical field intensity the corrections can dominate over the classical energy. Assuming $D = -0.3J$ and $B = 0.95\frac{-2D}{g\mu_B}$ we obtained energy of a single domain wall in arbitrary units as a

function of the elliptic parameter k (figures 5.4, 5.6 and 5.7), which is directly connected to the width of stripe domains, with blue line denoting classical energy and red line denoting total energy. In the case of $d = 2$ the quantum corrections are visibly causing a repelling force between domain walls, which will be more pronounced near the critical field intensity due to the established dependence of classical energy and quantum corrections on $\sqrt{-2D - g\mu_B B}$. In the $d = 3$ case the corrections have a minimal impact on the overall energy and seem to merely amplify the classical dependence on k parameter. The most interesting is the $d = 1$ case as was the case for single domain walls. From physical standpoint more important than the energy of a single domain wall will be the energy density (figure 5.5). As can be seen there is a well defined minimum, which means that, if we apply the corrections to the easy axis magnetic chain we can predict width of stripe domains, which minimises the energy density, as a function of magnetic field and material parameters. Such a dependence might be measurable in experiments. Furthermore, quantum energy seems to be much more dependent on the elliptic modulus than the classical energy, which implies that the interaction between domain walls cannot be properly modelled by a classical system.
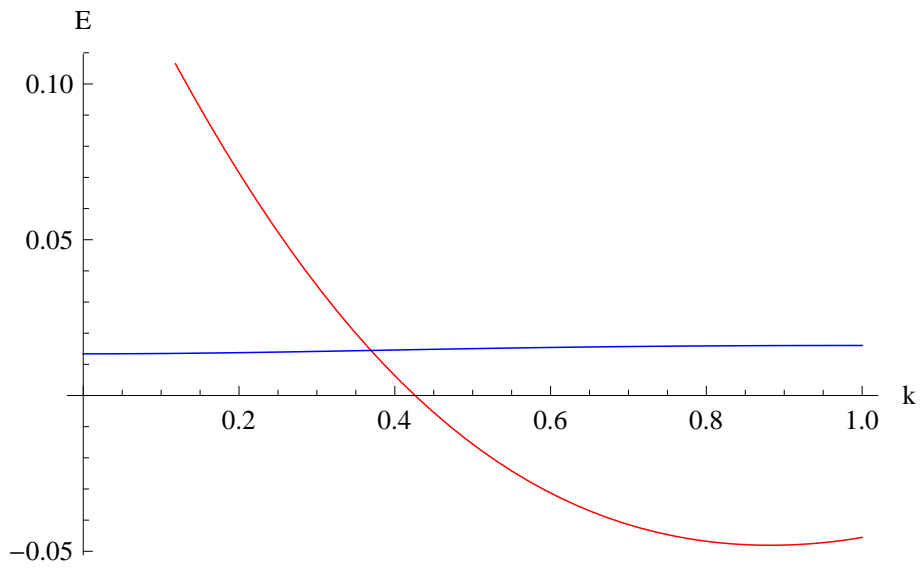
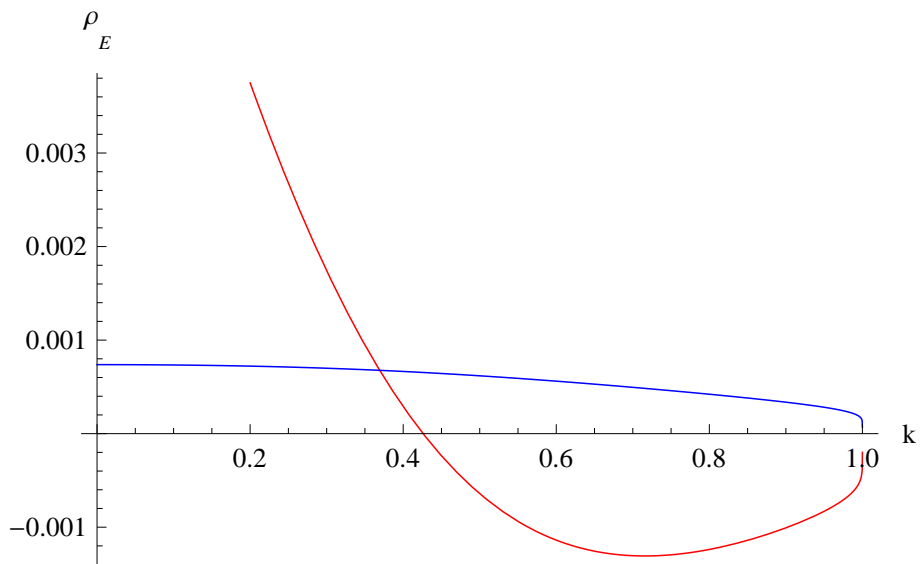Figure 5.4: Easy axis single domain wall energy for $d = 1$ in arbitrary units.

Figure 5.5: Easy axis average energy density for $d = 1$ in arbitrary units.
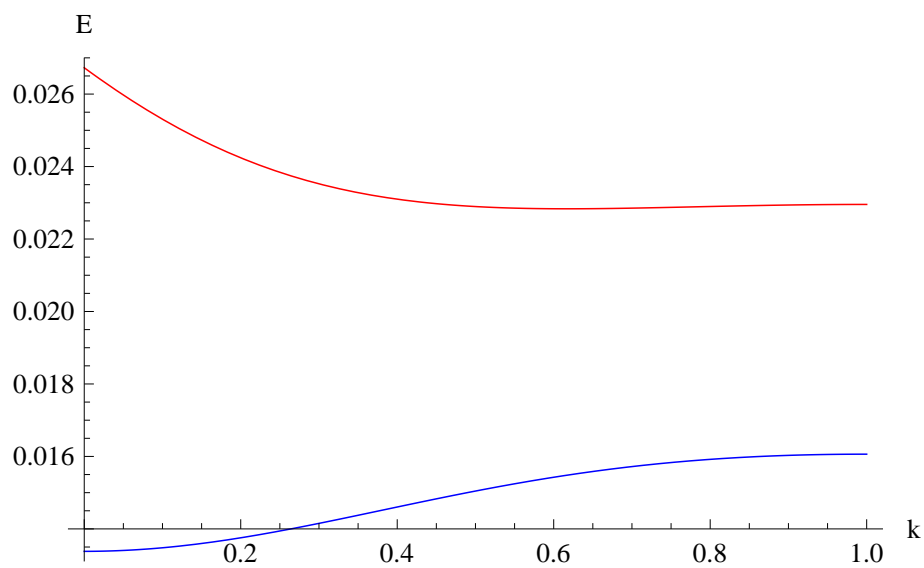
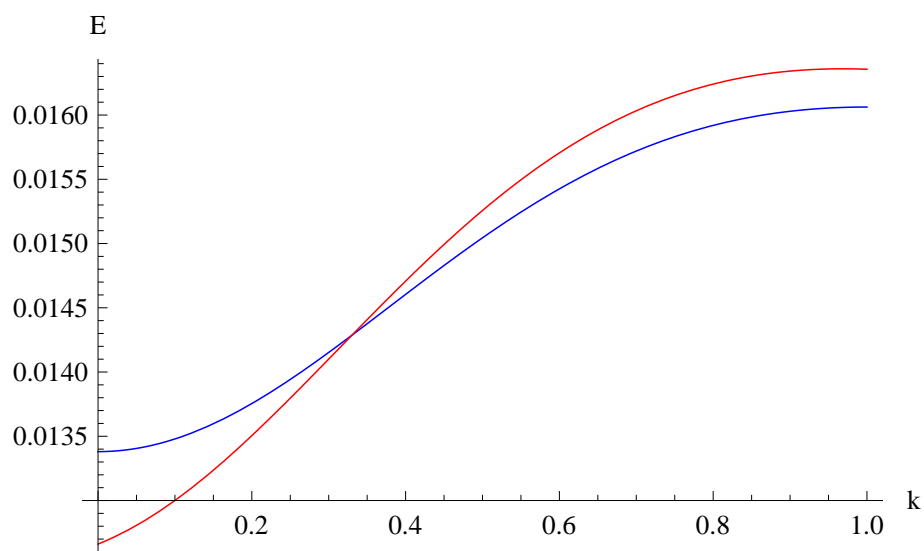Figure 5.6: Easy axis single domain wall energy for $d = 2$ in arbitrary units.

Figure 5.7: Easy axis single domain wall energy for $d = 3$ in arbitrary units.

Chapter 6

Conclusions

1. If we carry all physical constants through the calculations, it becomes evident, that quantum corrections to energy are independent of the energy scale of the classical system. If we were to multiply the classical action integral by any constant, it would not affect the corrections due to the way parameter A is cut by regularization. Even if our choice of the regularization coefficient r^2 was incorrect, scaling of the classical Hamiltonian would at most result in a logarithmic change in quantum corrections. In a way it coincides with the intuition, that quantum effects should only be noticeable in small scale systems.

2. The zeta-function regularization scheme is incomplete in the sense, that it does not give clear method of choosing the regularization coefficient r^2. The choice of vacuum cutoff in the case of fields over a finite interval is not necessarily straightforward as well. We can solve the problem by comparing the energy corrections for cases solved by other methods and extending the results to those otherwise unattainable as we did for the ϕ^4 and Sine-Gordon models.

3. Zeta-function regularization scheme does have the advantage of being easily extended to an arbitrary number of dimensions and the meaning of regularization steps can be directly linked to the definition of Maslov-Feynman integral. Moreover, Maslov approach to semiclassical quantization requires identical regularization in both quantum mechanics and quantum field theory, which might be useful in general studies of path integrals over fields.

4. In the case of easy axis domain walls, quantum corrections should dominate over classical energy in $1+1$ and $1+2$ cases leading to spontaneous domain creation in the $1+1$ case in a certain range of external magnetic fields.

5. Energy corrections show strong dependence on the overall number of dimensions of the classical system. It would be of interest to calculate energy corrections without the continuum approximation to research the system's geometry effect on energy in semiclassical regime.

6. In principle, Maslov approximation of path integrals can be used for non-trivialy multidimensional or dynamic fields as long as eigenvector of relevant operators form a base (see Section 3.2). We plan to use Darboux and Moutard to study such cases explicitly.

7. Algebraic analysis of Drach equation and resulting solving algorithm are of general enough nature, to allow a theoretical study of generalizations of Lamé operator to arbitrary polynomial potentials. It is also of interest to study the possibility of construction of Drach equation analogues for a wider class of ordinary differential equations.

The path integral formulation of quantum field theory used in this thesis is non-relativistic in the sense it is equivalent to the Schrödinger viewpoint.

Appendix A

Harmonic oscillator

Let us consider a classical harmonic oscillator

$$S(\phi) = \int_0^T \left[\frac{M}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 - \frac{k\phi^2}{2} \right] dt, \tag{A.0.1}$$

which after introduction of standard dimensionless variable $t = Tt'$ gives

$$S(\phi) = T \int_0^1 \left[\frac{M}{2T^2} \left(\frac{\partial \phi}{\partial t'} \right)^2 - \frac{k\phi^2}{2} \right] dt'. \tag{A.0.2}$$

We will now calculate the quantum propagator in the Maslov approximation 3.3

$$K(q, q', t, t') \approx e^{\frac{i}{\hbar} S(\phi)} \sqrt{\frac{\det L_0}{\det L}} \tag{A.0.3}$$

(with ϕ as a classical path between (q, t) and (q', t')), which is particularly easy, since the second derivative of the potential is trivial and does not depend on the classical solution

$$L = \frac{iM}{2T\pi\hbar r^2} \left(\frac{\partial^2}{\partial t'^2} - \frac{kT^2}{M} \right), \tag{A.0.4}$$

$$L_0 = \frac{iM}{2T\pi\hbar r^2} \left(\frac{\partial^2}{\partial t'^2} + C \right). \tag{A.0.5}$$

We intentionally leave the vacuum potential undecided to study its impact on the result. In order to obtain meaningful results for non-stationary solutions, we can't use continuum approximation of respective spectra. Therefore

$$\gamma_L(\tau) = \sum_{n \in \mathbb{Z}\{0\}} e^{-\frac{iM(\pi^2 n^2 - \frac{kT^2}{M})}{2T\pi\hbar r^2} \tau} \tag{A.0.6}$$

and

$$\gamma_{L_0}(\tau) = \sum_{n \in \mathbb{Z}\{0\}} e^{-\frac{iM(\pi^2 n^2 + C)}{2T\pi\hbar r^2} \tau}. \tag{A.0.7}$$

Let us now write down the ζ function

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \tau^{s-1}[\gamma_L(\tau) - \gamma_{L_0}(\tau)]d\tau. \tag{A.0.8}$$

We use the substitution $\tau_A = A\tau$ with $A = \frac{M}{2T\pi\hbar r^2}$

$$\zeta(s) = \frac{A^{-s}}{\Gamma(s)} \int_0^\infty \tau_A^{s-1}[\gamma_L(\tau_A) - \gamma_{L_0}(\tau_A)]d\tau_A. \tag{A.0.9}$$

After inserting the γ functions

$$\zeta(s) = \frac{A^{-s}}{\Gamma(s)} \int_0^\infty \tau_A^{s-1}(e^{i\frac{kT^2}{M}\tau_A} - e^{-iC\tau_A}) \sum_{n\in\mathbb{Z}} e^{-i\pi^2 n^2 \tau_A} d\tau_A. \tag{A.0.10}$$

The joining of the sums is acceptable, since they are unconditionally convergent. We can now perform the Mellin transform

$$\zeta(s) = A^{-s}i^{-s} \sum_{n\in\mathbb{Z}\{0\}} (\pi^2 n^2 - \frac{kT^2}{M})^{-s} - (\pi^2 n^2 + C)^{-s}. \tag{A.0.11}$$

Derivative at 0 gives us

$$\frac{\partial\zeta}{\partial s}(0) = -\ln(iA) + \sum_{n\in\mathbb{Z}\{0\}} \ln\left(\frac{\pi^2 n^2 + C}{\pi^2 n^2 - \frac{kT^2}{M}}\right). \tag{A.0.12}$$

From this form we can build the propagator (A.0.3)

$$e^{\frac{i}{\hbar}S(\phi)} \left(\sqrt{iA} \prod_{n\in\mathbb{Z}\{0\}} \sqrt{\frac{\pi^2 n^2 + C}{\pi^2 n^2 - \frac{kT^2}{M}}}\right), \tag{A.0.13}$$

which can be simplified to

$$e^{\frac{i}{\hbar}S(\phi)} \left(\sqrt{iA} \prod_{n\in\mathbb{Z}\{0\}} \sqrt{\frac{1 + \frac{C}{\pi^2 n^2}}{1 - \frac{kT^2}{\pi^2 n^2 M}}}\right) \tag{A.0.14}$$

and finally written down as

$$e^{\frac{i}{\hbar}S(\phi)} \sqrt{iA} \sqrt{\frac{kT^2}{CM}} \frac{\sinh\left(\sqrt{C}\right)}{\sin\left(\sqrt{\frac{k}{M}}T\right)}. \tag{A.0.15}$$

We arrive at an expression, which looks like a naive ordering of L and L_0 eigenvalues with an additional term $\sqrt{iA}$, which clearly needs to by regularized through a proper choice of base functions norm r, since the whole expression should converge to 1 for $T \to 0$. It is important to mention, that we can extract an arbitrary constant out of the sum in (A.0.11). For instance, we could have written

$$\zeta(s) = A^{-s}i^{-s}\pi^{-2s} \sum_{n\in\mathbb{Z}\{0\}} \left(n^2 - \frac{kT^2}{M\pi^2}\right)^{-s} - \left(n^2 + \frac{C}{\pi^2}\right)^{-s} \tag{A.0.16}$$

and arrive at

$$
e^{\frac{i}{\hbar}S(\phi)}\sqrt{iA\pi^2}\sqrt{\frac{kT^2}{CM}}\frac{\sinh\left(\sqrt{C}\right)}{\sin\left(\sqrt{\frac{k}{M}}T\right)}, \tag{A.0.17}
$$

which does not contradict (A.0.15), since the differences will be covered by the normalising factor. After inserting the value of A we obtain

$$
e^{\frac{i}{\hbar}S(\phi)}\sqrt{i\frac{\pi kT}{2\hbar r^2 C}}\frac{\sinh\left(\sqrt{C}\right)}{\sin\left(\sqrt{\frac{k}{M}}T\right)}. \tag{A.0.18}
$$

We can normalize the expression using the most important quality of the propagator

$$
K(q,q',t,t') = \int_{-\infty}^{\infty} K(q,q'',t,t'')K(q'',q',t'',t')dq''. \tag{A.0.19}
$$

It is important to note, that in this particular case we seem to have an overabundance of normalising factors, since we can use r (norm of the base functions used in Taylor expansion), the vacuum potential (which often is locked by the convergence condition) or the normalization of integral over q (which is directly available only in the quantum-mechanical case). For the sake of simplicity and without abandoning the generality of our calculations we will assume $C \to 0_+$ and keep the normalization of the integral as in (A.0.19). Using the expression for the action integral over a classical path between two arbitrary points in space-time

$$
S(q,q',t,t') = \frac{\sqrt{kM}}{2\sin\left(\sqrt{\frac{k}{M}}(t'-t)\right)}\left[(q^2+q'^2)\cos\left(\sqrt{\frac{k}{m}}(t'-t)\right) - 2qq'\right]. \tag{A.0.20}
$$

We can calculate the integral in (A.0.19) remembering that r can be a function of the time interval (in the following equations explicitly denoted as r(t))

$$
\frac{\sqrt{i\frac{\pi k(t'-t)}{2\hbar r^2(t'-t)}}}{\sin\left(\sqrt{\frac{k}{M}}(t'-t)\right)} = \sqrt{\frac{2i\hbar\pi \sin\left(\sqrt{\frac{k}{M}}(t'-t'')\right)\sin\left(\sqrt{\frac{k}{M}}(t''-t)\right)}{\sqrt{kM}\sin\left(\sqrt{\frac{k}{M}}(t'-t)\right)}}
$$
$$
\times \frac{\sqrt{i\frac{\pi k(t''-t)}{2\hbar r^2(t''-t)}}\sqrt{i\frac{\pi k(t'-t'')}{2\hbar r^2(t'-t'')}}}{\sin\left(\sqrt{\frac{k}{M}}(t'-t'')\right)\sin\left(\sqrt{\frac{k}{M}}(t''-t)\right)} \tag{A.0.21}
$$

with the $e^{\frac{i}{\hbar}S(\phi)}$ already cut on both sides for the purpose of clarity. After simplifying the expression we obtain

$$
\sqrt{\frac{i\pi k(t'-t)}{2\hbar r^2(t'-t)\sin\left(\sqrt{\frac{k}{M}}(t'-t)\right)}} = \times\sqrt{\frac{2i\hbar\pi}{\sqrt{kM}}}\sqrt{\frac{i\pi k(t''-t)}{2\hbar r^2(t''-t)\sin\left(\sqrt{\frac{k}{M}}(t''-t)\right)}}
$$
$$
\sqrt{\frac{i\pi k(t'-t'')}{2\hbar r^2(t'-t'')\sin\left(\sqrt{\frac{k}{M}}(t'-t'')\right)}}, \tag{A.0.22}
$$

which reveals a straightforward choice of r^2

$$\mathrm{r}^2(T) = -\frac{\pi^2 T \sqrt{k}}{\sqrt{M} \sin\left(\sqrt{\frac{k}{M}}T\right)} \tag{A.0.23}$$

and the final form of the propagation operator

$$K(q, q', t, t') = e^{\frac{i}{\hbar} \frac{\sqrt{kM}}{2 \sin\left(\sqrt{\frac{k}{M}}(t'-t)\right)} \left[(q^2+q'^2)\cos\left(\sqrt{\frac{k}{m}}(t'-t)\right) - 2qq'\right]} \sqrt{\frac{\sqrt{kM}}{2i\hbar\pi \sin\left(\sqrt{\frac{k}{M}}(t'-t)\right)}}. \tag{A.0.24}$$

It is of note, that in this case the semiclassical approach gives the exact result (compare with [116], where the harmonic oscillator propagator was obtained with three different methods). Additionally, it is worth noting, that the normalising factor r is not necessarily a simple function of the time interval and even the qualitative behavior of the propagator might not be fully visible before the normalization.

Appendix B

Mathematica code

Below, we present a sample implementation of the solving algorithm for the Drach equation described in Section 4.4.1.

```
Clear[U, L, m, z, k, r, n, H, P, PP, f, q]
U[z_] := m^2 (5 k^2 - 1) - 6 m^2 k^2 z ;
(*U is the chosen potential*)
L[z_] := 4 m^2 (1 - k^2) z + 4 m^2 (2 k^2 - 1) z^2 - 4 m^2 k^2 z^3;
(*L represents (z'_x)^2*)
kk = 0;
While[Not[D[U[z], {z, kk + 1}] === 0], kk++]
(*calculation of the order of polynomial U*)
n = r /. Solve[(r^2 + kk r)*D[L[z], {z, kk + 2}]/(kk + 2)! -
        4/(kk!)*D[U[z], {z, kk}] == 0, r][[2]];
(*calculation of the order of polynomial P*)
P[z_] := Sum[PP[i] z^i,{i,0,n}];
(*construction of polynomial P as an explicit sum in powers of z*)
f[z_] :=
 2 P[z] (D[P[z], {z, 2}] L[z] + D[L[z], z] D[P[z], z]/2) -
  L[z] (D[P[z], z])^2 - 4 (U[z] - p) P[z]^2
(*function f contains the whole left hand side of equation*)
Do[r = PP[i - n - kk] /.
    Solve[{D[f[z], {z, i}] == 0, z == 0}, PP[i - n - kk]];
  PP[i - n - kk] = r[[1]], {i, 2 n + kk - 1, n + kk, -1}];
(*evaluation of P_n (p)*)
If[Simplify[D[f[z], {z, 1}]] === 0, , "No solution"]
(*checking the solution*)
q = -f[z] /. z -> 0;
(*evaluation of Q (p) or more precisely 4Q (p)*)
FullSimplify[P[z]/Sqrt[q]]
(*presentation of results*)
```

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