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DOCTORAL DISSERTATION

Title of PhD dissertation: Field quantization in reducible representations of harmonic oscillator Lie algebras – application to relativistic EPR correlations of photons.

Title of PhD dissertation (in Polish): Kwantowanie pola w redukowalnych reprezentacjach algebr Lie oscylatora harmonicznego – zastosowanie do relatywistycznych korelacji EPR fotonów.

Supervisor   *signature*
Prof. dr hab. Marek Czachor

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Contents

Papers

This work includes results from the following papers:

1. M. Czachor, K. Wrzask, *Automatic regularization by quantization in reducible representations of CCR: Point-form quantum optics with classical sources*, Int. J. Theor. Phys. 48, 2511 (2009); e-print arXiv:math-ph/0806.3510v3.

2. K. Wrzask, *Four-dimensional photon polarization space in the background of reducible representation algebras*; e-print arXiv:quant-ph/1511.00515.

3. K. Wrzask, *Relativistic EPR-type experiments for photons in the background of reducible representation HOLA algebras*; e-print arXiv:quant-ph/1511.02658.

It was a sort of act of faith with us that any questions which describe fundamental laws of nature must have great mathematical beauty in them. It was a very profitable religion to hold and can be considered as the basis of much of our success.

Dirac

Streszczenie

Główną motywacją pracy jest przyjrzenie się relatywistycznemu modelowi pól fotonowych w zaproponowanych przez Marka Czachora redukowalnych reprezentacjach algebr Lie oscylatora harmonicznego i zastosowanie takiego modelu do relatywistycznych korelacji EPR fotonów. Zaprezentowano czterowymiarową przestrzeń polaryzacji fotonów taką, która w porównaniu z formalizmem Gupty-Bleulera daje inną interpretację operatorów kreacji i anihilacji dla „czasowego" stopnia swobody. Taka interpretacja, wywodząca się z konstrukcji kowariantnego Hamiltonianu, daje dodatnio zdefiniowane normy dla wszystkich czterech polaryzacyjnych stopni swobody. Pokazano stany, które produkują standardowe pola elektromagnetyczne (tzn. fotony z dwiema polaryzacjami) z czterowymiarowych kowariantnych pól (tzn. pól z dwiema dodatkowymi polaryzacjami: „podłużnej" i „czasowej"). Ponadto został zaproponowany model czterech stanów Bella, taki w którym korelacje pozostają maksymalne dla dwóch baz: liniowej i kołowej we wszystkich układach odniesienia. Na końcu zostały wyliczone relatywistyczne korelacje EPR dla dwóch przypadków: gdy na oba detektory działa transformacja Lorentza w taki sposób, że pozostają w tym samym układzie odniesienia oraz w przypadku gdy tylko jeden z detektorów pozostaje pod działaniem tejże transformacji.

Summary

The main motivation for this work was to take a closer look at a relativistic model for photon fields in reducible representations of harmonic oscillator Lie algebras proposed by Marek Czachor with an application to relativistic EPR-type correlations. A four-dimensional photon polarization space, such that gives a different interpretation of the ladder operators for the time-like degree comparing with the Gupta-Bleuler formulation is presented. This interpretation, coming from a construction of a covariant Hamiltonian, gives positive defined norms for all the four polarization degrees of freedom. Further states that reproduce standard electromagnetic fields (i.e. photons with two polarization degrees of freedom) from the four-dimensional covariant formalism (i.e. with two additional longitudinal and time-like polarization degrees of freedom) are shown explicitly and discussed. A model for the four Bell states, such that maintains maximally correlated in two polarization bases: linear and circular in all reference frames is developed. Finally relativistic correlation functions are derived for two cases: when the detectors are transformed under Lorentz transformation in such a way that they still remain in the same reference frame and when just one of the detectors is transformed.

Introduction

Relativistic EPR-type experiments have been discussed at least from 1997 mainly in theoretical background [53]-[100]. This is a good example of a problem where quantum mechanics and theory of relativity are treated under one roof and an occasion to take a closer look at a relativistic model for photon fields in reducible representation of harmonic oscillator Lie algebras (HOLA) proposed by Czachor [6]-[19]. Some of the difficulties for Lorentz transformation law of Bell states, such as the dependence of Wigner rotations on momentum, covariance of the potential operator and a model of an invariant two-photon field, are investigated here in the background of reducible representations. On the other hand, employing a model for relativistic EPR-type experiments may show the role that the oscillator number N and vacuum probability density $Z(\boldsymbol{k})$, known from reducible representations, play in this model.

Therefore, this thesis is organized as follows. In chapter 1 spinor geometry is introduced starting from the Minkowski vector space coming from the stereographic projection. Next in this chapter basic spinor algebra and abstract Penrose index notation are introduced.

In chapter 2 the reducible representations of harmonic oscillator Lie algebras are introduced. We start with basic motivation for such a quantization, introducing spectral decomposition of a frequency operator. Next, the Hamiltonian, ladder and number operators are introduced for the reducible one-oscillator ($N = 1$) representation. It is shown that the reducible representations taken within the whole frequency spectrum have the "standard-theory" harmonic oscillator Lie algebra. Further in this chapter an extension to arbitrary N-oscillator is shown, discussing the ladder and number operators for such a representation. Of special importance for the reducible representation quantization is the definition of vacuum, because regularization is a consequence of employing a special form of vacuum with a probability density function $Z(\boldsymbol{k})$.

In chapter 3 a construction coming from a covariant Hamiltonian for a four-dimensional oscillator is shown. When constructing such a four-dimensional oscillator, one should consider what is the consequence of creating particles on the energy of the whole system, i.e. does it raise the energy level or lower it? This is discussed first for space-like polarization degrees of freedom and then two different interpretations of a time-like polarization degree are considered. It turns out that, when assuming for time-like photons the energy spectrum bounded from the top, we can preserve the positive norms as needed for the probability interpretation of quantum mechanics.

In chapter 4 we will combine the two previous formalisms, i.e. the reducible representation and the four-dimensional polarization space, into one. First, a construction of the four-dimensional oscillator for the reducible representation is presented. Then vacuum and the potential operator for this model are introduced. From this a question comes up: how does such a theory, with four polarization degrees of freedom, correspond with Maxwell electromagnetism theory. It turns out that there exists a vector space of states that reproduces standard Maxwell electrodynamics. Next, the potential operator and the electromagnetic field operator for this representation are introduced. Finally, we discuss coherent states and show that the Ψ_{EM} states have a coherent-like structure.

In chapter 5 Lorentz transformations coming form SL(2,C) transformations defined on the spin-frame level are introduced. It turns out that on this level Lorentz transformation is accompanied by another symmetry which also keeps the spin-frame condition. On the tetrad level this symmetry manifests itself as a gauge transformation. Further in this chapter reducible representations of Lorentz and gauge transformations are introduced, with generators that start from four-dimensional canonical variables. Next, transformation properties of the potential operator, electromagnetic field operator and vacuum are presented, following a discussion of the gauge transformation. This chapter closes with an introduction to four-translations.

In chapter 6 a two-photon field operator for the reducible N-oscillator representations is discussed. Further in this chapter a model for all four Bell states is proposed.

In chapter 7 transformation properties of two-photon fields are discussed. A model for scalar fields is proposed assuming that the four Bell states are maximally correlated or anti-correlated in two polarization bases, and that the polarization angle is momentum dependent.

In chapter 8 observables for EPR-type experiments are calculated. First a yes-no observable describing the measurement in detectors is introduced. Then a correlation function for the EPR-type experiment is calculated for a two-photon state. Next, the EPR correlation function is treated separately for Bell states maximally anti-correlated or correlated in circular polarizations.

Finally, chapter 9 deals with EPR experiments under Lorentz transformations. Two cases are considered: when two detectors are transformed under a Lorentz transformation in such a way that they still remain in the same reference frame, and when only one of the detectors is transformed under the Lorentz transformation.

This work closes with a series of mathematical appendices.

1 Spinor space-time geometry and algebra

This chapter is preliminary. In this work Minkowski and null tetrads play a role of photon polarization vectors. The Penrose abstract index notation is not that popular, therefore we start by introducing the Minkowski vector space and the stereographic projection in section 1.1. Next, some basic spinor algebra and abstract Penrose index notations are explained in sections 1.2 and 1.3 respectively.

1.1 Minkowski vector space and the stereographic projection

A Minkowski vector space is a four-dimensional vector space V over a field of real numbers $\mathbb{R}$. It is equipped with a symmetric bilinear form with signature $(+,-,-,-)$. Let us consider a basis of linear independent vectors $t,x,y,z \in V$ so that any $U \in V$ can be uniquely expressed in the form

$$U = U^0 t + U^1 x + U^2 y + U^3 z, \tag{1}$$

where the coordinates of U are denoted by U^0, U^1, U^2, U^3. There is a special case

$$U = Tt + Xx + Yy + Zz, \tag{2}$$

of null vectors, or vectors on the light cone having coordinates such that

$$T^2 - X^2 - Y^2 - Z^2 = 0. \tag{3}$$

An introduction to spinors is usually made with a method called stereographic projection. To present this method let us start with a unit vector with three components (x,y,z):

$$x^2 + y^2 + z^2 \quad = \quad 1. \tag{4}$$

This describes a unit sphere in R^3 with polar coordinates

$$\begin{cases} x &= \sin\theta\cos\phi, \\ y &= \sin\theta\sin\phi, \\ z &= \cos\theta. \end{cases} \tag{5}$$

Points on this sphere can be also described by a stereographic projection on the plane $z=0$. A useful form can be obtained if this plane is parameterized by complex coordinates ζ linked to a point on the sphere by:

$$\zeta \;=\; \frac{x+iy}{1-z} \;=\; e^{i\phi}\operatorname{ctg}\frac{\theta}{2}. \tag{6}$$

For example, the north pole of the sphere with coordinates $(x=0, y=0, z=1)$ corresponds to $\zeta=\infty$, and the south pole $(0,0,-1)$ to $\zeta=0$. Another parametrization of a sphere is also possible

$$\zeta = \frac{\xi}{\eta}, \tag{7}$$

where

$$\begin{cases} \xi &= \sqrt{r}e^{i(\alpha+\phi)/2}\cos\frac{\theta}{2}, \\ \eta &= \sqrt{r}e^{i(\alpha-\phi)/2}\sin\frac{\theta}{2}. \end{cases} \tag{8}$$

Now the north and south poles of the sphere correspond to $(\xi,0)$ and $(0,\eta)$ respectively. In this parametrization there is an extra degree of freedom since (ξ,η) and say $(\lambda\xi,\lambda\eta)$, $\lambda\in\mathbb{C}$, describe the same point ζ. The four components $(1,x,y,z)$ of a null future-pointing world-vector are linked to those coordinates by

$$(1,x,y,z) \;=\; \left(1,\; \frac{\xi\bar{\eta}+\eta\bar{\xi}}{\xi\bar{\xi}+\eta\bar{\eta}},\; -i\frac{\xi\bar{\eta}-\eta\bar{\xi}}{\xi\bar{\xi}+\eta\bar{\eta}},\; \frac{\xi\bar{\xi}-\eta\bar{\eta}}{\xi\bar{\xi}+\eta\bar{\eta}}\right). \tag{9}$$

Multiplying both sides by $(\xi\bar{\xi}+\eta\bar{\eta})\sqrt{2}$ we get a link between complex numbers (ξ,η) and coordinates of a future-pointing null vector i.e.

$$(T,X,Y,Z) \;=\; \frac{1}{\sqrt{2}}\left(\xi\bar{\xi}+\eta\bar{\eta},\; \xi\bar{\eta}+\eta\bar{\xi},\; -i(\xi\bar{\eta}-\eta\bar{\xi}),\; \xi\bar{\xi}-\eta\bar{\eta}\right). \tag{10}$$

As one can see the change of phase in (8) leaves the components of the four-vector unchanged. So the aim now is to associate with (ξ, η) a richer geometrical structure that reduces the redundancy to a single sign ambiguity. This structure will be a "null flag", i.e. the previous null vector associated with a half null plane attached to the vector $\boldsymbol{K}$, which will represent the phase and will be called a "flag plane".

1.2 Spinor algebra

Now let us associate the two complex numbers (ξ, η) with a spin-vector $\boldsymbol{\psi}$ such that

$$\xi = \psi^0, \tag{11}$$
$$\eta = \psi^1. \tag{12}$$

Two spin-vectors, ω_A and π_A, satisfying

$$\omega_A \pi^A = 1, \qquad \pi_A \omega^A = -1. \tag{13}$$

are called a spin-frame. From the anti-symmetry of the inner product we have

$$\omega_A \omega^A = 0, \qquad \pi_A \pi^A = 0. \tag{14}$$

Now any spin-vector can be written in terms of a spin-frame as

$$\psi^A = \psi^0 \omega^A + \psi^1 \pi^A, \tag{15}$$

and the components of a spin-vector are

$$\psi^0 = -\pi_A \psi^A, \qquad \psi^1 = \omega_A \psi^A. \tag{16}$$

In this notation A, B are abstract indices which means they are just labels that do not take any numerical value, and the boldface ones $\boldsymbol{A}, \boldsymbol{B}$ will take numerical values $0, 1$. The abstract spinor index is a capital Roman letter, either primed $A', B', ...$, or unprimed $A, B, ...$ In the literature, sometimes instead of a prime a dot is used, for an example $\dot{A}$. The spin space is a two dimensional symplectic complex vector space. An element of this space is written with an unprimed superscript, for example ψ^A. The symplectic form is denoted by ε_{AB} and is a skew-symmetric complex bilinear form, i.e. $\varepsilon_{AB} = -\varepsilon_{BA}$. The action of the bilinear form on vectors $\varepsilon_{AB}\psi^A\phi^B$ is a complex number. The element of the dual spin space is written with an unprimed subscript ψ_A. The dual symplectic form is ε_{AB} such that $\varepsilon_{AB}\varepsilon^{CB} = \varepsilon_A{}^C$. It is often convenient to use a collective symbol $\varepsilon_{\boldsymbol{A}}{}^A$ for a spin basis such that

$$\varepsilon_0{}^A = \omega^A, \qquad \varepsilon_1{}^A = \pi^A. \tag{17}$$

Furthermore, the components of ε_{AB} with respect to the spin-frame basis are

$$\varepsilon_{\boldsymbol{AB}} = \varepsilon_{AB}\varepsilon_{\boldsymbol{A}}{}^A \varepsilon_{\boldsymbol{B}}{}^B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{18}$$

A dual basis denoted by $\varepsilon_A{}^{\boldsymbol{A}}$ must then satisfy

$$\varepsilon_{\boldsymbol{A}}{}^A \varepsilon_A{}^{\boldsymbol{B}} = \varepsilon_{\boldsymbol{A}}{}^{\boldsymbol{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{19}$$

It thus implies that

$$\varepsilon_A{}^0 = -\pi_A, \quad \varepsilon_A{}^1 = \omega_A. \tag{20}$$

Therefore, the spin-frame and the dual spin-frame can be written in a matrix notation as

$$\varepsilon_A{}^{\boldsymbol{A}} = \begin{pmatrix} -\pi_A \\ \omega_A \end{pmatrix}, \tag{21}$$
$$\varepsilon_{\boldsymbol{A}}{}^A = \begin{pmatrix} \omega^A \\ \pi^A \end{pmatrix}. \tag{22}$$

Any spin-frame satisfies the following formulas:

$$\varepsilon^{AB} = \omega^A \pi^B - \pi^A \omega^B, \qquad \varepsilon_{AB} = \omega_A \pi_B - \pi_A \omega_B, \qquad \varepsilon_A{}^B = \omega_A \pi^B - \pi_A \omega^B, \tag{23}$$

$$\varepsilon_{AB} = -\varepsilon_{BA}, \qquad \varepsilon^{AB} = -\varepsilon^{BA}, \qquad \varepsilon_A{}^B = -\varepsilon^B{}_A = \delta^B_A, \tag{24}$$

$$\varepsilon_{AB}\varepsilon^{CB} = \varepsilon_A{}^C = \delta^C_A, \qquad \varepsilon^{AB}\varepsilon_{AC} = \varepsilon_C{}^B = \delta^B_C, \tag{25}$$

$$\varepsilon_{AB}\varepsilon_C{}^D + \varepsilon_{BC}\varepsilon_A{}^D + \varepsilon_{CA}\varepsilon_B{}^D = 0, \tag{26}$$

$$\varepsilon_{AB}\varepsilon_{CD} + \varepsilon_{BC}\varepsilon_{AD} + \varepsilon_{CA}\varepsilon_{BD} = 0. \tag{27}$$

1.3 Penrose abstract indices notation for tetrads

Three types of indices will be introduced here: the boldface indices $\boldsymbol{a}$, $\boldsymbol{b}$ take numerical values 0, 1, 2, 3 and are related to a concrete choice of basis. The italics a, b are abstract indices and specify types of tensor objects. The abstract index formalism allows to work at a basis independent level, with all the operations on indices we know from the matrix calculus.

The Minkowski space has signature $(+,-,-,-)$ and the metric tensor is denoted by g_{ab}, g^{ab}. $g_{\boldsymbol{ab}}$ and $g^{\boldsymbol{ab}}$ are matrices diag$(+,-,-,-)$. A Minkowski tetrad $g_a{}^{\boldsymbol{a}}$, indexed by indices that are partly boldfaced and partly italic, consists of four four-vectors $g_a{}^0$, $g_a{}^1$, $g_a{}^2$, $g_a{}^3$. Two types of tetrads will be employed. The momentum independent tetrad $g_a{}^{\boldsymbol{a}}$ satisfies $k^0 = |\boldsymbol{k}| = k^a g_a{}^0$, $k^1 = k^a g_a{}^1$, $k^2 = k^a g_a{}^2$, $k^3 = k^a g_a{}^3$, and defines decomposition into energy and momentum in the Lorentz invariant measure

$$d\Gamma(\boldsymbol{k}) = \frac{d^3k}{(2\pi)^3 2|\boldsymbol{k}|}. \tag{28}$$

The four momentum $k^a = k^a(\boldsymbol{k})$ can be written in a spinor notation as $k^a(\boldsymbol{k}) = \pi^A(\boldsymbol{k})\pi^{A'}(\boldsymbol{k})$, where $\pi^A(\boldsymbol{k})$ is a spinor field defined by $k^a(\boldsymbol{k})$ up to a phase factor. For any $\pi^A(\boldsymbol{k})$ there exists another spinor field $\omega^A(\boldsymbol{k})$ satisfying the spin-frame condition $\omega_A(\boldsymbol{k})\pi^A(\boldsymbol{k}) = 1$.

We also consider a general field of tetrads defined on the light cone and denoted by $g_a{}^{\boldsymbol{a}}(\boldsymbol{k})$. Here $g_a{}^1(\boldsymbol{k})$ and $g_a{}^2(\boldsymbol{k})$ can play a role of transverse polarization vectors and $g_a{}^3(\boldsymbol{k})$ is parallel to the 3-momentum. Indices a and $\boldsymbol{a}$ can be raised and lowered by means of the Minkowski metric tensor g_{ab}, g^{ab}, $g_{\boldsymbol{ab}}$ and $g^{\boldsymbol{ab}}$.

The null tetrad will be indexed by indices that are partly boldfaced-primed and partly italic $g^a{}_{\boldsymbol{b'}}$. It is important to distinguish between $\boldsymbol{a}$ and $\boldsymbol{AA'}$, and we will employ the convention where $\boldsymbol{a'} = 00', 01', 10', 11'$ and $g_{\boldsymbol{a'b'}} = \varepsilon_{\boldsymbol{AB}}\varepsilon_{\boldsymbol{A'B'}}$, $g^{\boldsymbol{a'b'}} = \varepsilon^{\boldsymbol{AB}}\varepsilon^{\boldsymbol{A'B'}}$. We raise and lower indices $\boldsymbol{AA'}$ by means of $\varepsilon_{\boldsymbol{AB}}\varepsilon_{\boldsymbol{A'B'}}$ and indices $\boldsymbol{a'}$ by means of the matrix

$$g_{\boldsymbol{a'b'}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{29}$$

The null tetrad associated with spin-frames can be written as

$$g^a{}_{\boldsymbol{b'}} = \begin{pmatrix} g^a{}_{00'} \\ g^a{}_{01'} \\ g^a{}_{10'} \\ g^a{}_{11'} \end{pmatrix} = \begin{pmatrix} \varepsilon^A{}_0\varepsilon^{A'}{}_{0'} \\ \varepsilon^A{}_0\varepsilon^{A'}{}_{1'} \\ \varepsilon^A{}_1\varepsilon^{A'}{}_{0'} \\ \varepsilon^A{}_1\varepsilon^{A'}{}_{1'} \end{pmatrix} = \begin{pmatrix} \omega^A\omega^{A'} \\ \omega^A\pi^{A'} \\ \pi^A\omega^{A'} \\ \pi^A\pi^{A'} \end{pmatrix} = \begin{pmatrix} \omega^a \\ m^a \\ \bar{m}^a \\ k^a \end{pmatrix} \tag{30}$$

and dually

$$g_a{}^{\boldsymbol{b'}} = \begin{pmatrix} g_a{}^{00'} \\ g_a{}^{01'} \\ g_a{}^{10'} \\ g_a{}^{11'} \end{pmatrix} = \begin{pmatrix} \varepsilon_A{}^0\varepsilon_{A'}{}^{0'} \\ \varepsilon_A{}^0\varepsilon_{A'}{}^{1'} \\ \varepsilon_A{}^1\varepsilon_{A'}{}^{0'} \\ \varepsilon_A{}^1\varepsilon_{A'}{}^{1'} \end{pmatrix} = \begin{pmatrix} \pi_A\pi_{A'} \\ -\pi_A\omega_{A'} \\ -\omega_A\pi_{A'} \\ \omega_A\omega_{A'} \end{pmatrix} = \begin{pmatrix} k_a \\ -\bar{m}_a \\ -m_a \\ \omega_a \end{pmatrix}. \tag{31}$$

Here $g^a{}_{01'}(\boldsymbol{k})$ and $g^a{}_{10'}(\boldsymbol{k})$ can play the role of circular photon polarization vectors.

There is a relation between a Minkowski tetrad, indexed by indices that are partly boldfaced and partly italic, and a null tetrad

$$g^a{}_{\boldsymbol{a}}(\boldsymbol{k}) \;=\; g_{\boldsymbol{a}\boldsymbol{b}'}g^{a\boldsymbol{b}'}(\boldsymbol{k}) \;=\; g_{\boldsymbol{a}}{}^{\boldsymbol{b}'}g^a{}_{\boldsymbol{b}'}(\boldsymbol{k}) \;=\; g_{\boldsymbol{a}}{}^{\boldsymbol{B}\boldsymbol{B}'}g^a{}_{\boldsymbol{B}\boldsymbol{B}'}(\boldsymbol{k}), \tag{32}$$

$$\begin{aligned} g^a{}_{\boldsymbol{a}} \;&=\; \begin{pmatrix} g^a{}_0 \\ g^a{}_1 \\ g^a{}_2 \\ g^a{}_3 \end{pmatrix} \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \omega^A\omega^{A'} \\ \omega^A\pi^{A'} \\ \pi^A\omega^{A'} \\ \pi^A\pi^{A'} \end{pmatrix} \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} \omega^a + k^a \\ m^a + \bar{m}^a \\ im^a - i\bar{m}^a \\ \omega^a - k^a \end{pmatrix} \\ &=\; \begin{pmatrix} t^a \\ x^a \\ y^a \\ z^a \end{pmatrix} \end{aligned} \tag{33}$$

and dually we can write

$$g_a{}^{\boldsymbol{a}}(\boldsymbol{k}) \;=\; g^{\boldsymbol{a}\boldsymbol{b}'}g_{a\boldsymbol{b}'}(\boldsymbol{k}) \;=\; g^{\boldsymbol{a}}{}_{\boldsymbol{b}'}g_a{}^{\boldsymbol{b}'}(\boldsymbol{k}) \;=\; g^{\boldsymbol{a}}{}_{\boldsymbol{B}\boldsymbol{B}'}g_a{}^{\boldsymbol{B}\boldsymbol{B}'}(\boldsymbol{k}), \tag{34}$$

$$\begin{aligned} g_a{}^{\boldsymbol{a}} \;&=\; \begin{pmatrix} g_a{}^0 \\ g_a{}^1 \\ g_a{}^2 \\ g_a{}^3 \end{pmatrix} \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -i & i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \pi_A\pi_{A'} \\ -\pi_A\omega_{A'} \\ -\omega_A\pi_{A'} \\ \omega_A\omega_{A'} \end{pmatrix} \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} k_a + \omega_a \\ -\bar{m}_a - m_a \\ i\bar{m}_a - im_a \\ k_a - \omega_a \end{pmatrix} \\ &=\; \begin{pmatrix} t_a \\ -x_a \\ -y_a \\ -z_a \end{pmatrix}. \end{aligned} \tag{35}$$

Here the gs with the partly boldfaced and partly boldfaced-primed indices are the Infeld-van der Waerden symbols which can be written in the following matrix forms

$$g^{\boldsymbol{a}}{}_{\boldsymbol{b}'} \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -i & i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \tag{36}$$

$$g_{\boldsymbol{a}}{}^{\boldsymbol{b}'} \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}. \tag{37}$$

These Infeld-van der Waerden symbols, in their matrix forms, in Penrose abstract index formalism, are used to translate formulas into matrix forms. As an example it is easy to show that

$$g_{\boldsymbol{a}'\boldsymbol{b}'} \;=\; g_{\boldsymbol{a}'}{}^{\boldsymbol{a}}g_{\boldsymbol{a}\boldsymbol{b}'} \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & i & -i & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \;=\; \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{38}$$

To perform matrix manipulation for (38) we need to transpose (36) and lower the index $\boldsymbol{b}'$ in (37) by means of the matrix (29). Also for the metric tensor we can show that

$$g_{\boldsymbol{a}\boldsymbol{b}} \;=\; g_{\boldsymbol{a}}{}^{\boldsymbol{a}'}g_{\boldsymbol{a}'\boldsymbol{b}} \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & i & 0 \\ 0 & -1 & -i & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{39}$$

The relation between the Minkowski tetrad and the metric tensor g_{ab} is

$$g_{ab} \;=\; g_a{}^{\boldsymbol{a}}(\boldsymbol{k}) g_b{}^{\boldsymbol{b}}(\boldsymbol{k}) g_{\boldsymbol{ab}} \;=\; -\,x_a(\boldsymbol{k})x_b(\boldsymbol{k}) - y_a(\boldsymbol{k})y_b(\boldsymbol{k}) - z_a(\boldsymbol{k})z_b(\boldsymbol{k}) + t_a(\boldsymbol{k})t_b(\boldsymbol{k}). \tag{40}$$

Analogously, the relation between the null tetrad and the metric tensor g_{ab} is

$$g_{ab} \;=\; g_a{}^{\boldsymbol{a}'}(\boldsymbol{k}) g_b{}^{\boldsymbol{b}'}(\boldsymbol{k}) g_{\boldsymbol{a}'\boldsymbol{b}'} \;=\; k_a(\boldsymbol{k})\omega_b(\boldsymbol{k}) + \omega_a(\boldsymbol{k})k_b(\boldsymbol{k}) - m_a(\boldsymbol{k})\bar{m}_b(\boldsymbol{k}) - \bar{m}_a(\boldsymbol{k})m_b(\boldsymbol{k}). \tag{41}$$

2 Reducible representations

For further analysis let us assume the convention $c = \hbar = 1$. When building a relativistic model for photons one has to consider photon's momentum and polarization. In this mathematical model these two quantities will be described in a tensor product structure, i.e.

$$\text{photon momentum space} \otimes \text{photon polarization space} \tag{42}$$

It also should be stressed that in relativistic context spin and momentum are not independent degrees of freedom. This will be discussed further in chapter 5. Some preliminary aspects of the reducible representation will be discussed in the present chapter. Such a model has strong arguments in it's favor, mostly because it naturally handles some of the infrared and ultraviolet divergences. In 2.1 the basic idea of the model is introduced. Further, in 2.2 and 2.3, ladder and number operators are introduced for $N = 1$ oscillator space of the theory. How the extension to N-oscillator space looks like and what is the definition of the number operator in N space are discussed in 2.4 and 2.5. Finally how to represent vacuum is shown in 2.6.

2.1 Motivation

In 1925 Heisenberg, Born and Jordan observed that energies of classical free fields look in Fourier space like those of oscillator ensembles. It should be stressed that at that time Heisenberg, Born and Jordan did not know the notation of a Fock space and may not fully understood the role of eigenvalues of operators. Having to consider oscillators with different frequencies they considered one oscillator for each frequency mode. The ensemble had to be infinite since the number of modes was infinite.

It is a well known problem that standard canonical procedures for field quantization result in various infinities. It was shown in [6] by Czachor that the assumption of having one oscillator for each frequency mode may not be natural. This thought continued in a series of papers on reducible representation of CCR [7] - [18], also a draft of lecture notes may be found [19]. The main idea for reducible representations is that each of the oscillators is a wave packet, a superposition of infinitely many different momentum states.

To describe this concept in more detail, let us first introduce a spectral decomposition of a frequency operator

$$\Omega \;=\; \int d\Gamma(\boldsymbol{k})\ \omega(\boldsymbol{k})\ |\boldsymbol{k}\rangle\langle\boldsymbol{k}|, \tag{43}$$

so that (43) fulfills the eigenvalue problem

$$\Omega\ |\boldsymbol{k}\rangle \;=\; \int d\Gamma(\boldsymbol{k}')\ \omega(\boldsymbol{k}')\ |\boldsymbol{k}'\rangle\langle\boldsymbol{k}'|\boldsymbol{k}\rangle \;=\; \omega(\boldsymbol{k})\ |\boldsymbol{k}\rangle. \tag{44}$$

Here $d\Gamma(\boldsymbol{k})$ in the Lorentz invariant measure (28). Furthermore, kets of momentum are normalized to

$$\langle\boldsymbol{k}|\boldsymbol{k}'\rangle \;=\; (2\pi)^3 2|\boldsymbol{k}|\delta^{(3)}(\boldsymbol{k},\boldsymbol{k}') \;=\; \delta_\Gamma(\boldsymbol{k},\boldsymbol{k}') \tag{45}$$

and the resolution of unity is

$$\int_{R^3} \frac{d^3k}{(2\pi)^3 2|\boldsymbol{k}|}\ |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \;=\; \int_{R^3} d\Gamma(\boldsymbol{k})\ |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \;=\; I. \tag{46}$$

The energy for photons, assuming the convention $\hbar = 1$, is $E(\boldsymbol{k}) = \omega(\boldsymbol{k}) = |\boldsymbol{k}|$. So the simplest Hamiltonian for one kind of polarization can be written in the form

$$H \;=\; \Omega \otimes \left(a^\dagger a + \frac{1}{2}\right) \;=\; \int d\Gamma(\boldsymbol{k})\ \omega(\boldsymbol{k})\ |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes \left(a^\dagger a + \frac{1}{2}\right), \tag{47}$$

so that

$$H|\boldsymbol{k}, n\rangle \;=\; \omega(\boldsymbol{k}) \left(n + \frac{1}{2}\right) |\boldsymbol{k}, n\rangle. \tag{48}$$

Here $|n\rangle$ is the eigenvector of a "standard-theory" number operator $a^\dagger a$, where $[a, a^\dagger] = 1$. Now, let us introduce an operator that lives in both momentum and polarization spaces

$$a(\boldsymbol{k}, 1) = |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes a. \tag{49}$$

Using the resolution of unity, we can also define an operator within the whole spectrum of frequencies

$$a(1) \;=\; \int d\Gamma(\boldsymbol{k})\; a(\boldsymbol{k}, 1) \;\;=\;\; \int d\Gamma(\boldsymbol{k})\; |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes a \;=\; I \otimes a, \tag{50}$$

such that the basic commutator is $[a(1), a^\dagger(1)] = I \otimes 1$. Here 1 in bracket of $a(1)$ denotes that it is an $N = 1$ oscillator representation.

2.2 Creation and annihilation operators

Let us start from the $N = 1$ oscillator space with two polarization degrees of freedom. Then the Hilbert space $\mathcal{H}(1)$ of one oscillator is spanned by

$$|\boldsymbol{k}, n_1, n_2\rangle \;=\; |\boldsymbol{k}\rangle \otimes |n_1\rangle \otimes |n_2\rangle. \tag{51}$$

Here kets $|n_\alpha\rangle$ are eigenvectors of $a^\dagger_\alpha a_\alpha$. At this point let us consider a two dimensional polarization oscillator and do not determine what kind of polarizations these dimensions determine, only that $a_\alpha, a_{\alpha'}$ satisfy canonical commutation relations (CCR) typical for irreducible representation, i.e. $[a_\alpha, a^\dagger_{\alpha'}] = \delta_{\alpha,\alpha'},\ \alpha, \alpha' = 1, 2$. Although in the next chapter relativistically covariant four-dimensional polarizations are introduced, in the present chapter we will use only two dimensions just to fully concentrate on the subject of reducible representation. Now the reducible representation of the ladder operators is

$$\begin{aligned} a_\alpha(\boldsymbol{k}, 1) &\;=\; |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes a_\alpha, & (52)\\ a_\alpha(\boldsymbol{k}, 1)^\dagger &\;=\; |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes a^\dagger_\alpha. & (53)\end{aligned}$$

The parameter 1 in the argument of $a(\boldsymbol{k}, 1)$ in (52) and (53) indicates that this is an $N = 1$ oscillator representation. Sub-index α stands for one of the two possible polarization dimensions of an oscillator, where

$$a_1 \;=\; \mathrm{a}_1 \otimes 1, \qquad a_2 \;=\; 1 \otimes \mathrm{a}_2. \tag{54}$$

Then the commutation relations for the reducible representation of creation and annihilation operators are

$$[a_\alpha(\boldsymbol{k}, 1), a_{\alpha'}(\boldsymbol{k}', 1)^\dagger] \;=\; \delta_{\alpha,\alpha'}\delta_\Gamma(\boldsymbol{k}, \boldsymbol{k}')|\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes 1_2, \tag{55}$$

where 1_2 denotes that it is a tensor product of two 1s. This representation is reducible since the right-hand side of the commutator is an operator valued distribution with

$$I(\boldsymbol{k}, 1) \;=\; |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes 1 \otimes 1 \;=\; |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes 1_2 \tag{56}$$

belonging to the center of algebra, i.e. it commutes with the ladder operators

$$[a_\alpha(\boldsymbol{k}, 1), I(\boldsymbol{k}', 1)] \;=\; [a_\alpha(\boldsymbol{k}, 1)^\dagger, I(\boldsymbol{k}', 1)] \;=\; 0. \tag{57}$$

Furthermore, the operator $I(\boldsymbol{k}, 1)$ forms the resolution of unity for $\mathcal{H}(1)$ Hilbert space:

$$\int d\Gamma(\boldsymbol{k})\; I(\boldsymbol{k}, 1) \;=\; I(1). \tag{58}$$

2.3 CRR and HOLA algebra in $N = 1$ space representation

The number operator for the reducible representation in $\mathcal{H}(1)$ Hilbert space will be defined as

$$n_\alpha(\boldsymbol{k}, 1) \;=\; |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes a^\dagger_\alpha a_\alpha. \tag{59}$$

Let us note that this definition is not equivalent to

$$n'_\alpha(\boldsymbol{k},1) \;=\; a_\alpha(\boldsymbol{k},1)^\dagger a_\alpha(\boldsymbol{k},1) \;=\; \delta_\Gamma(\boldsymbol{k},\boldsymbol{k}) n_\alpha(\boldsymbol{k},1). \tag{60}$$

The definition of Dirac delta $\delta_\Gamma(\boldsymbol{k},\boldsymbol{k})$ may not be ambiguous [19]. We can also define a number operator within the whole spectrum of frequencies as follows

$$n_\alpha(1) \;=\; a_\alpha(1)^\dagger a_\alpha(1) \;=\; I\otimes a_\alpha^\dagger a_\alpha \;=\; \int d\Gamma(\boldsymbol{k})|\boldsymbol{k}\rangle\langle\boldsymbol{k}|\otimes a_\alpha^\dagger a_\alpha \;=\; \int d\Gamma(\boldsymbol{k}) n(\boldsymbol{k},1). \tag{61}$$

So the eigenvalue definition of the number operator, i.e. "how many photons are there with α polarization within the whole frequency spectrum" would be

$$n_\alpha(1)|\boldsymbol{k},n_1,n_2\rangle \;=\; \int d\Gamma(\boldsymbol{k}') n_\alpha(\boldsymbol{k}',1)|\boldsymbol{k},n_1,n_2\rangle \;=\; n_\alpha|\boldsymbol{k},n_1,n_2\rangle. \tag{62}$$

Now the following Lie algebra for the reducible representation holds

$$[a_\alpha(\boldsymbol{k},1),a_{\alpha'}(\boldsymbol{k}',1)^\dagger] \;=\; \delta_{\alpha,\alpha'}\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}')I(\boldsymbol{k},1), \tag{63}$$
$$[a_\alpha(\boldsymbol{k},1),n_{\alpha'}(\boldsymbol{k}',1)] \;=\; \delta_{\alpha,\alpha'}\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}')a_\alpha(\boldsymbol{k},1), \tag{64}$$
$$[a_\alpha(\boldsymbol{k},1)^\dagger,n_{\alpha'}(\boldsymbol{k}',1)] \;=\; -\delta_{\alpha,\alpha'}\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}')a_\alpha(\boldsymbol{k},1)^\dagger. \tag{65}$$

Furthermore, for the representation within the whole frequency spectrum we have

$$[a_\alpha(1),a_{\alpha'}(1)^\dagger] \;=\; \delta_{\alpha,\alpha'}I(1), \tag{66}$$
$$[a_\alpha(1),n_{\alpha'}(1)] \;=\; \delta_{\alpha,\alpha'}a_\alpha(1), \tag{67}$$
$$[a_\alpha(1)^\dagger,n_{\alpha'}(1)] \;=\; -\delta_{\alpha,\alpha'}a_\alpha(1)^\dagger. \tag{68}$$

As one can see, the Lie algebra for the whole frequency spectrum has the "standard-theory" structure.

2.4 Creation and annihilation operators in N-oscillator space representations

Now let us discuss an extension of the theory to N-oscillators. The parameter N characterizes the reducible representation. This parameter is not related to the number of photons. The Hilbert space for N-oscillators reads

$$\mathcal{H}(N) \;=\; \underbrace{\mathcal{H}(1)\otimes\ldots\otimes\mathcal{H}(1)}_{N} \;=\; \mathcal{H}^{\otimes N}, \tag{69}$$

and is spanned by kets of the form

$$|\boldsymbol{k}_1,n_1^1,n_2^1\rangle\otimes|\boldsymbol{k}_2,n_1^2,n_2^2\rangle\otimes\ldots\otimes|\boldsymbol{k}_N,n_1^N,n_2^N\rangle = \bigotimes_{m=1}^{N}|\boldsymbol{k}_m,n_1^m,n_2^m\rangle. \tag{70}$$

Let us also define an operator

$$A^{(n)} \;=\; \underbrace{I\otimes...\otimes I}_{n-1}\otimes A\otimes\underbrace{I\otimes...\otimes I}_{N-n}. \tag{71}$$

The upper index (n) shows the "position" of the A operator in $\mathcal{H}(N)$ space. Operator (71) has the following properties:

$$[A^{(n)},B^{(m)}] \;=\; [A,B]^{(n)}\delta_{n,m}, \tag{72}$$
$$\left[\sum_n^N A^{(n)},\sum_m^N B^{(m)}\right] \;=\; \sum_n^N [A,B]^{(n)}, \tag{73}$$
$$(A+B)^{(n)} \;=\; A^{(n)}+B^{(n)}, \tag{74}$$
$$(AB)^{(n)} \;=\; A^{(n)}B^{(n)}, \tag{75}$$
$$(A^{(n)})^m \;=\; (A^m)^{(n)}, \tag{76}$$
$$\exp\left(\sum_{n=1}^N A^{(n)}\right) \;=\; \big(\exp A\big)^{\otimes N}. \tag{77}$$

A natural extension of creation and annihilation operators in reducible representation to the N-oscillator space is

$$a_\alpha(\boldsymbol{k},N) = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} a_\alpha(\boldsymbol{k},1)^{(n)} = \frac{1}{\sqrt{N}}\Big(a_\alpha(\boldsymbol{k},1)\otimes ...\otimes I(1) + ... + I(1)\otimes ...\otimes a_\alpha(\boldsymbol{k},1)\Big), \tag{78}$$

$$a_\alpha(\boldsymbol{k},N)^\dagger = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} a_\alpha(\boldsymbol{k},1)^{\dagger(n)} = \frac{1}{\sqrt{N}}\Big(a_\alpha(\boldsymbol{k},1)^\dagger\otimes ...\otimes I(1) + ... + I(1)\otimes ...\otimes a_\alpha(\boldsymbol{k},1)^\dagger\Big). \tag{79}$$

The factor $\frac{1}{\sqrt{N}}$ is the normalization factor for N-oscillator representation. The CCR algebras still hold (see (I.1) in the appendix):

$$[a_\alpha(\boldsymbol{k},N), a_{\alpha'}(\boldsymbol{k}',N)^\dagger] = \delta_{\alpha,\alpha'}\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}')I(\boldsymbol{k},N), \tag{80}$$

where at the right-hand side we have an operator

$$I(\boldsymbol{k},N) = \frac{1}{N}\sum_{n=1}^{N} I(\boldsymbol{k},1)^{(n)} = \frac{1}{N}\Big(I(\boldsymbol{k},1)\otimes ...\otimes I(1) + ... + I(1)\otimes ...\otimes I(\boldsymbol{k},1)\Big), \tag{81}$$

which for all N is also in the center of the algebra since

$$[a_\alpha(\boldsymbol{k},N), I(\boldsymbol{k}',N)] = [a_\alpha(\boldsymbol{k},N)^\dagger, I(\boldsymbol{k}',N)] = 0. \tag{82}$$

$I(\boldsymbol{k},N)$ satisfy the resolution of unity for the Hilbert space $\mathcal{H}(N)$:

$$\int d\Gamma(\boldsymbol{k})\; I(\boldsymbol{k},N) = I(N) = \underbrace{I(1)\otimes ...\otimes I(1)}_{N}. \tag{83}$$

Let us also define the ladder operators within the whole spectrum of frequencies, i.e.

$$a_\alpha(N) = \frac{1}{\sqrt{N}}\sum_{n=1}^{N}\int d\Gamma(\boldsymbol{k})\; a_\alpha(\boldsymbol{k},1)^{(n)} = \frac{1}{\sqrt{N}}\Big(a_\alpha(1)\otimes ...\otimes I(1) + ... + I(1)\otimes ...\otimes a_\alpha(1)\Big), \tag{84}$$

$$a_\alpha(N)^\dagger = \frac{1}{\sqrt{N}}\sum_{n=1}^{N}\int d\Gamma(\boldsymbol{k})\; a_\alpha(\boldsymbol{k},1)^{\dagger(n)} = \frac{1}{\sqrt{N}}\Big(a_\alpha(1)^\dagger\otimes ...\otimes I(1) + ... + I(1)\otimes ...\otimes a_\alpha(1)^\dagger\Big). \tag{85}$$

Then CCR algebra for the whole frequency spectrum operators is satisfied,

$$[a_\alpha(N), a_{\alpha'}(N)^\dagger] = \delta_{\alpha,\alpha'}I(N). \tag{86}$$

Again the representation within the whole frequency spectrum keeps the "standard-theory" structure.

2.5 Number operator in N-oscillator representations

The structure of the ladder operators for N-oscillators (78) and (79), together with the reducible representation, creates possibilities for several definitions of the number operator. Following the lecture notes [19] three possibilities will be discussed here. First let us consider a product of two reducible-representation ladder operators for N-oscillators, i.e.

$$\begin{aligned}
n^I_\alpha(\boldsymbol{k},N) &= a_\alpha(\boldsymbol{k},N)^\dagger a_\alpha(\boldsymbol{k},N) = \frac{1}{N}\sum_{m,n=1}^{N} a_\alpha(\boldsymbol{k},1)^{\dagger(m)}a_\alpha(\boldsymbol{k},1)^{(n)}\\
&= \frac{1}{N}\Big(a_\alpha(\boldsymbol{k},1)^\dagger\otimes ...\otimes I(1) + ... + I(1)\otimes ...\otimes a_\alpha(\boldsymbol{k},1)^\dagger\Big)\\
&\times \Big(a_\alpha(\boldsymbol{k},1)\otimes ...\otimes I(1) + ... + I(1)\otimes ...\otimes a_\alpha(\boldsymbol{k},1)\Big)\\
&= \frac{1}{N}\Big(n'_\alpha(\boldsymbol{k},1)\otimes ...\otimes I(1) + ... + I(1)\otimes ...\otimes n'_\alpha(\boldsymbol{k},1)\Big) + n'_{\mathrm{int},\alpha}(\boldsymbol{k},N).
\end{aligned} \tag{87}$$

For the second choice let us use the number operator (60) for $N = 1$ representation and write:

$$\begin{aligned} n^{II}_\alpha(\boldsymbol{k}, N) &= \sum_{m=1}^{N} \left(a_\alpha(\boldsymbol{k},1)^\dagger a_\alpha(\boldsymbol{k},1)\right)^{(m)} = \sum_{m=1}^{N} n'_\alpha(\boldsymbol{k},1)^{(m)} \\ &= n'_\alpha(\boldsymbol{k},1) \otimes ... \otimes I(1) + ... + I(1) \otimes ... \otimes n'_\alpha(\boldsymbol{k},1). \end{aligned} \tag{88}$$

Finally using the definition (59) of the number operator for $N = 1$ representation we can write:

$$\begin{aligned} n^{III}_\alpha(\boldsymbol{k}, N) &= \sum_{m=1}^{N} \left(|\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes a^\dagger_\alpha a_\alpha\right)^{(m)} = \sum_{m=1}^{N} n_\alpha(\boldsymbol{k},1)^{(m)} \\ &= n_\alpha(\boldsymbol{k},1) \otimes ... \otimes I(1) + ... + I(1) \otimes ... \otimes n_\alpha(\boldsymbol{k},1). \end{aligned} \tag{89}$$

The first choice generates additional $N(N-1)$ interaction terms. Second and third would be equivalent if $\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}) = 1$. Now let us take a closer look at the Lie algebras with respect to the choice of n-s:

$$[a_\alpha(\boldsymbol{k}, N), n^{I}_{\alpha'}(\boldsymbol{k}', N)] = \delta_{\alpha,\alpha'}\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}')a_\alpha(\boldsymbol{k}, N)I(\boldsymbol{k}, N), \tag{90}$$

$$[a_\alpha(\boldsymbol{k}, N), n^{II}_{\alpha'}(\boldsymbol{k}', N)] = \delta_{\alpha,\alpha'}\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}')\delta_\Gamma(\boldsymbol{k},\boldsymbol{k})a_\alpha(\boldsymbol{k}, N), \tag{91}$$

$$[a_\alpha(\boldsymbol{k}, N), n^{III}_{\alpha'}(\boldsymbol{k}', N)] = \delta_{\alpha,\alpha'}\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}')a_\alpha(\boldsymbol{k}, N). \tag{92}$$

So the issue of the choice of representation of the number operator boils down to the two questions: what is the definition of $\delta_\Gamma(\boldsymbol{k},\boldsymbol{k})$, and how does $I(\boldsymbol{k}, N)$ in (90) contribute to the theory?

At this stage of constructing the formalism it is worth checking the Lie algebras for operators within the whole frequency spectrum

$$[a_\alpha(N), n^{I}_{\alpha'}(N)] = \delta_{\alpha,\alpha'}\int d\Gamma(\boldsymbol{k})a_\alpha(\boldsymbol{k}, N)I(\boldsymbol{k}, N), \tag{93}$$

$$[a_\alpha(N), n^{II}_{\alpha'}(N)] = \delta_{\alpha,\alpha'}\int d\Gamma(\boldsymbol{k})\delta_\Gamma(\boldsymbol{k},\boldsymbol{k})a_\alpha(\boldsymbol{k}, N), \tag{94}$$

$$[a_\alpha(N), n^{III}_{\alpha'}(N)] = \delta_{\alpha,\alpha'}a_\alpha(N). \tag{95}$$

We can see that only the choice (95) maintains the standard structure of HOLA.

Another digression on this subject should be made. Looking at the number operator for $N = 1$ oscillators within the whole frequency spectrum (61) one may want to extend this definition to all N and consider

$$\begin{aligned} n^{IV}_\alpha(N) &= a_\alpha(N)^\dagger a_\alpha(N) = \int d\Gamma(\boldsymbol{k})\int d\Gamma(\boldsymbol{k}')a_\alpha(\boldsymbol{k}, N)^\dagger a_\alpha(\boldsymbol{k}', N) \\ &= \int d\Gamma(\boldsymbol{k})\int d\Gamma(\boldsymbol{k}')\frac{1}{N}\sum_{m,n=1}^{N} a_\alpha(\boldsymbol{k},1)^{\dagger(m)}a_\alpha(\boldsymbol{k}',1)^{(n)}. \end{aligned} \tag{96}$$

This operator has the same commutation relations as in (95), however the problem is that there is no unique definition for the reducible representation, i.e.

$$a_\alpha(\boldsymbol{k}, N)^\dagger a_\alpha(N) \neq a_\alpha(N)^\dagger a_\alpha(\boldsymbol{k}, N), \tag{97}$$

that is the commutation relations for both cases would be

$$[a_\alpha(\boldsymbol{k}, N), a_{\alpha'}(\boldsymbol{k}', N)^\dagger a_{\alpha'}(N)] = \delta_{\alpha,\alpha'}\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}')I(\boldsymbol{k}, N)a_\alpha(N), \tag{98}$$

$$[a_\alpha(\boldsymbol{k}, N), a_{\alpha'}(N)^\dagger a_{\alpha'}(\boldsymbol{k}', N)] = \delta_{\alpha,\alpha'}I(\boldsymbol{k}, N)a_\alpha(\boldsymbol{k}', N). \tag{99}$$

Due to this inconsistency, (96) will not be considered any more.

Although at this point n^{III} (89) looks like the best choice for the number operator in reducible representation, we will also check the eigenvalue problem, i.e.

$$n^{III}_\alpha(N)\bigotimes_{m=1}^{N}|\boldsymbol{k}_m, n_1^m, n_2^m\rangle = \sum_{m'=1}^{N} n^{III}_\alpha(1)^{(m')}\bigotimes_{m=1}^{N}|\boldsymbol{k}_m, n_1^m, n_2^m\rangle = n_\alpha\bigotimes_{m=1}^{N}|\boldsymbol{k}_m, n_1^m, n_2^m\rangle, \tag{100}$$

where $n_\alpha = \sum_{m=1}^{N} n_\alpha^m$ is the total number of α polarized excitations in the N-oscillator system. From now on, when referring to the number operator for N-oscillator reducible representations, we will have in mind the formula (89) and the superscript III in $n^{III}(\boldsymbol{k}, N)$ will be dropped.

2.6 Vacuum

A vacuum for $N = 1$ reducible representation is any vector of the form

$$|O(1)\rangle \;=\; \int d\Gamma(\boldsymbol{k}) O(\boldsymbol{k}) |\boldsymbol{k}, 0, 0\rangle. \tag{101}$$

This definition implies that vacuum is any vector annihilated by all annihilation operators, i.e.

$$a_\alpha(1)|O(1)\rangle \;=\; 0. \tag{102}$$

From the normalization condition

$$\langle O(1)|O(1)\rangle \;=\; 1 \tag{103}$$

we get

$$\int d\Gamma(\boldsymbol{k}) |O(\boldsymbol{k})|^2 \;=\; \int d\Gamma(\boldsymbol{k}) Z(\boldsymbol{k}) \;=\; 1. \tag{104}$$

Here the scalar field $Z(\boldsymbol{k}) = |O(\boldsymbol{k})|^2$ represents vacuum probability density. Furthermore, square integrability of (104) implies that $Z(\boldsymbol{k})$ must decay at infinity. Moreover, $Z(\boldsymbol{k})$ is required to go to zero at $\boldsymbol{k} = 0$ in order to avoid infrared divergences [12]. It should be stressed that this point is of special importance for such a quantization. It turns out that regularization can be a consequence of employing such special form of scalar field in the definition of vacuum.

An extension to N-oscillator space is assumed to be a tensor product of $N = 1$ vacuum states, i.e.

$$|O(N)\rangle \;=\; |O(1)\rangle^{\otimes N} \;=\; \underbrace{|O(1)\rangle \otimes ... \otimes |O(1)\rangle}_{N}. \tag{105}$$

In analogy to (102), an N-oscillator vacuum can be defined as any vector annihilated by N-oscillator representation of annihilation operators

$$a_\alpha(N)|O(N)\rangle \;=\; 0. \tag{106}$$

Of course, the normalization condition for N representation still holds, i.e.

$$\langle O(N)|O(N)\rangle \;=\; \langle O(1)|O(1)\rangle^N \;=\; 1. \tag{107}$$

2.7 Conclusions and results

Most of the results from this section were presented in [6]–[13], and in lecture notes [19]. N-oscillator representations form several possibilities of defining a number operator. It has been shown that reducible representations of HOLA integrated over the whole frequency spectrum maintain the "standard theory" form of harmonic-oscillator Lie algebra if an appropriate definition (89) of the number operator is accepted. The eigenvalue problem for the whole frequency spectrum provides further arguments in favor of the definition (89).

3 Four-dimensional photon polarization space

Four-dimensional quantization, often called a Gupta-Bleuler type, is well known from the literature [30] - [36]. Except for the two "standard theory" transverse polarizations, two additional ones are introduced. These additional degrees of freedom are called time-like and longitudinal photons respectively. There seems to be no experiment verifying existence of such particles, therefore they are often called unphysical ore even "ghosts". Typically it is assumed that time-like photon states have negative norm. Unfortunately states with such nonpositive norms form a serious difficulty for the probability interpretation of quantum mechanics.

In this chapter we will introduce a four-dimensional covariant formalism for the photon polarization space with a different interpretation of the time-like polarization. The same four-dimensional quantization, as presented here, was already formulated by Czachor and Naudts [12] and further Czachor and Wrzask [13], where in the definition of the potential operator, in the place of the annihilation operator of Gupta-Bleuler-type potential for the time-like degree of freedom, stands a creation operator. In this chapter an interpretation of the time-like polarization of such quantization will be profoundly investigated. This analysis is a new result. Further the in next chapter it turns out that the contribution to gauge-invariant quantities of such time-like fields cancels against the longitudinal ones and only the two transverse polarization fields remain.

This chapter is organized as follows. In section 3.1 a construction coming from a covariant Hamiltonian for a four-dimensional oscillator is shown. When constructing such a four-dimensional oscillator one should consider what is the consequence of creating particles on the energy of the whole system, i.e. does it raise the energy level or lower it? This is discussed further in sections: 3.2 for the space-like polarization degrees of freedom and in 3.3 and 3.4 two different interpretations of the time-like polarization degree are considered. It turns out that, assuming for the time-like photons the energy spectrum bounded from the top and particles with negative energy, we can preserve the positive norms as needed for the probability interpretation of quantum mechanics.

3.1 Construction of four-dimensional polarization space

To construct a four-dimensional photon polarization space let us first introduce an abstract covariant Hamiltonian of the form

$$H = -\frac{p_{\boldsymbol{a}}p^{\boldsymbol{a}}}{2} - \frac{q_{\boldsymbol{a}}q^{\boldsymbol{a}}}{2}, \quad \boldsymbol{a} = 0,1,2,3. \tag{108}$$

Here $p_{\boldsymbol{a}}$ and $q_{\boldsymbol{a}}$ are some canonical variables such that

$$p_{\boldsymbol{a}} = i\partial_{\boldsymbol{a}} = i\frac{\partial}{\partial q^{\boldsymbol{a}}} = ig_{\boldsymbol{ab}}\frac{\partial}{\partial q_{\boldsymbol{b}}}, \tag{109}$$

with commutation relations

$$[q_{\boldsymbol{a}}, p_{\boldsymbol{b}}] = -ig_{\boldsymbol{ab}}. \tag{110}$$

Let us remind ourselves that the metric here is chosen as diag$(+,-,-,-)$. These canonical variables should not be mistaken with the position and momentum of the photon field. This construction is made strictly for the four degrees of photon polarization. Now let us define non-hermitian operators

$$a_{\boldsymbol{a}} = \frac{q_{\boldsymbol{a}} + ip_{\boldsymbol{a}}}{\sqrt{2}} = \frac{q_{\boldsymbol{a}} - \partial_{\boldsymbol{a}}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(q_0 - \frac{\partial}{\partial q_0},\ q_1 + \frac{\partial}{\partial q_1},\ q_2 + \frac{\partial}{\partial q_2},\ q_3 + \frac{\partial}{\partial q_3}\right), \tag{111}$$

$$a^\dagger_{\boldsymbol{a}} = \frac{q_{\boldsymbol{a}} - ip_{\boldsymbol{a}}}{\sqrt{2}} = \frac{q_{\boldsymbol{a}} + \partial_{\boldsymbol{a}}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(q_0 + \frac{\partial}{\partial q_0},\ q_1 - \frac{\partial}{\partial q_1},\ q_2 - \frac{\partial}{\partial q_2},\ q_3 - \frac{\partial}{\partial q_3}\right), \tag{112}$$

which satisfy the following commutation relations

$$[a_{\boldsymbol{a}}, a^\dagger_{\boldsymbol{b}}] = -g_{\boldsymbol{ab}}, \tag{113}$$

$$[a_{\boldsymbol{a}}, a_{\boldsymbol{b}}] = [a^\dagger_{\boldsymbol{a}}, a^\dagger_{\boldsymbol{b}}] = 0, \tag{114}$$

$$[a_{\boldsymbol{a}}, \left(a^\dagger_{\boldsymbol{b}}\right)^n] = \sum_{k=0}^{n-1} \left(a^\dagger_{\boldsymbol{b}}\right)^k [a_{\boldsymbol{a}}, a^\dagger_{\boldsymbol{b}}] \left(a^\dagger_{\boldsymbol{b}}\right)^{n-k-1} = -g_{\boldsymbol{ab}} n \left(a^\dagger_{\boldsymbol{b}}\right)^{n-1}, \tag{115}$$

$$[(a_{\boldsymbol{a}})^n, a^\dagger_{\boldsymbol{b}}] = \sum_{k=0}^{n-1} (a_{\boldsymbol{a}})^k [a_{\boldsymbol{a}}, a^\dagger_{\boldsymbol{b}}] (a_{\boldsymbol{a}})^{n-k-1} = -g_{\boldsymbol{ab}} n (a_{\boldsymbol{a}})^{n-1}. \tag{116}$$

Following J. Ch. Pain's paper [38], we can also write these in a very useful form

$$\left[(a_{\boldsymbol{a}})^n, \left(a^\dagger_{\boldsymbol{b}}\right)^m\right] = -\sum_{k=1}^{\min(n,m)} \frac{(g_{\boldsymbol{ab}})^k n! m!}{k!(n-k)!(m-k)!} (a^\dagger_{\boldsymbol{b}})^{m-k} (a_{\boldsymbol{a}})^{n-k}, \tag{117}$$

$$\left[(a_{\boldsymbol{a}})^n, \left(a^\dagger_{\boldsymbol{b}}\right)^m\right] = \sum_{k=1}^{\min(n,m)} \frac{(-g_{\boldsymbol{ab}})^k n! m!}{k!(n-k)!(m-k)!} (a_{\boldsymbol{a}})^{n-k} (a^\dagger_{\boldsymbol{b}})^{m-k}. \tag{118}$$

Furthermore, (111) and (112) can be easily inverted to give the canonical variables

$$p_{\boldsymbol{a}} = \frac{a_{\boldsymbol{a}} - a^\dagger_{\boldsymbol{a}}}{i\sqrt{2}}, \tag{119}$$

$$q_{\boldsymbol{a}} = \frac{a_{\boldsymbol{a}} + a^\dagger_{\boldsymbol{a}}}{\sqrt{2}}. \tag{120}$$

As a consequence of such a covariant formalism, there are four polarization degrees of freedom. They can be written in a four-dimensional tensor product space

$$a_1 = \mathrm{a}_1 \otimes 1 \otimes 1 \otimes 1, \quad a_2 = 1 \otimes \mathrm{a}_2 \otimes 1 \otimes 1, \quad a_3 = 1 \otimes 1 \otimes \mathrm{a}_3 \otimes 1, \quad a_0 = 1 \otimes 1 \otimes 1 \otimes \mathrm{a}_0, \tag{121}$$

where the four-dimensional space will be spanned by kets of the form

$$|n_1, n_2, n_3, n_0\rangle = |n_1\rangle \otimes |n_2\rangle \otimes |n_3\rangle \otimes |n_0\rangle. \tag{122}$$

Now let us take a closer look at

$$\begin{aligned} a^\dagger_{\boldsymbol{a}} a^{\boldsymbol{a}} &= \frac{1}{2}\left(q_{\boldsymbol{a}} - ip_{\boldsymbol{a}}\right)\left(q^{\boldsymbol{a}} + ip^{\boldsymbol{a}}\right) \\ &= \frac{1}{2}\left(q_{\boldsymbol{a}}q^{\boldsymbol{a}} + iq_{\boldsymbol{a}}p^{\boldsymbol{a}} - ip_{\boldsymbol{a}}q^{\boldsymbol{a}} + p_{\boldsymbol{a}}p^{\boldsymbol{a}}\right) \\ &= \frac{1}{2}q_{\boldsymbol{a}}q^{\boldsymbol{a}} + i\frac{1}{2}[q_{\boldsymbol{a}}, p^{\boldsymbol{a}}] + \frac{1}{2}p_{\boldsymbol{a}}p^{\boldsymbol{a}} \\ &= \frac{1}{2}q_{\boldsymbol{a}}q^{\boldsymbol{a}} + 2 + \frac{1}{2}p_{\boldsymbol{a}}p^{\boldsymbol{a}}. \end{aligned} \tag{123}$$

This implies that the Hamiltonian (108) can be written in terms of operators $a^\dagger_{\boldsymbol{a}}, a^{\boldsymbol{a}}$ in the form

$$H = -a^\dagger_{\boldsymbol{a}} a^{\boldsymbol{a}} + 2, \qquad \boldsymbol{a} = 0, 1, 2, 3. \tag{124}$$

Furthermore, the following commutation relations hold

$$[H, a_{\boldsymbol{a}}] = [-a^\dagger_{\boldsymbol{b}} a^{\boldsymbol{b}}, a_{\boldsymbol{a}}] = -a_{\boldsymbol{a}}, \tag{125}$$

$$[H, a^\dagger_{\boldsymbol{a}}] = [-a^\dagger_{\boldsymbol{b}} a^{\boldsymbol{b}}, a^\dagger_{\boldsymbol{a}}] = a^\dagger_{\boldsymbol{a}}. \tag{126}$$

Let us assume that the eigenvalue of the covariant Hamiltonian operator (108) acting on four-dimensional space is denoted by E.

$$H|n_1, n_2, n_3, n_0\rangle = E|n_1, n_2, n_3, n_0\rangle. \tag{127}$$

At this point this is really a quantity corresponding with the number of particles, but in the upcoming section 4, an extension of this model to reducible representation is made and $E(1)$ will correspond to the total (free-field) energy operator. Now it can be shown that indeed $a_{\boldsymbol{a}}$ lowers and $a^\dagger_{\boldsymbol{a}}$ raises E by 1, i.e.

$$\begin{aligned} Ha_{\boldsymbol{a}}|n_1,n_2,n_3,n_0\rangle &= [H,a_{\boldsymbol{a}}]|n_1,n_2,n_3,n_0\rangle + a_{\boldsymbol{a}}H|n_1,n_2,n_3,n_0\rangle \\ &= -a_{\boldsymbol{a}}|n_1,n_2,n_3,n_0\rangle + a_{\boldsymbol{a}}E|n_1,n_2,n_3,n_0\rangle \\ &= (E-1)a_{\boldsymbol{a}}|n_1,n_2,n_3,n_0\rangle, \end{aligned} \tag{128}$$

$$\begin{aligned} Ha^\dagger_{\boldsymbol{a}}|n_1,n_2,n_3,n_0\rangle &= [H,a^\dagger_{\boldsymbol{a}}]|n_1,n_2,n_3,n_0\rangle + a^\dagger_{\boldsymbol{a}}H|n_1,n_2,n_3,n_0\rangle \\ &= a^\dagger_{\boldsymbol{a}}|n_1,n_2,n_3,n_0\rangle + a^\dagger_{\boldsymbol{a}}E|n_1,n_2,n_3,n_0\rangle \\ &= (E+1)a^\dagger_{\boldsymbol{a}}|n_1,n_2,n_3,n_0\rangle. \end{aligned} \tag{129}$$

3.2 Construction for space-like photons

From the previous section we learn that (128) and (129) will define lowering and raising energy operators respectively and the raising operators will be denoted with a dagger. These operators are also known in the literature as annihilation and creation operators, but at this point this terminology will not be used for a reason. It will turn out that the definition of the 0 polarization degree of freedom may be ambiguous.

It is quite evident that for the polarization degrees of freedom $\boldsymbol{a} = 1,2,3$, the raising energy operators create new states. Let us then define vacuum states for these three photon polarization dimensions as normalized states that are annihilated by lowering-energy operators:

$$\mathrm{a}_{\boldsymbol{j}}|0\rangle = 0, \quad \boldsymbol{j}=1,2,3. \tag{130}$$

Now let us normalize these space-like states to 1. The state of $n_{\boldsymbol{j}}$ excitations must be proportional to n raising energy operators acting on ground state, so we can write

$$|n_{\boldsymbol{j}}\rangle \sim \mathrm{a}^{\dagger n}_{\boldsymbol{j}}|0\rangle. \tag{131}$$

Then the scalar product can be denoted as

$$\begin{aligned} \langle n_{\boldsymbol{i}}|n_{\boldsymbol{j}}\rangle \sim \langle 0|(\mathrm{a}_{\boldsymbol{i}})^n(\mathrm{a}^\dagger_{\boldsymbol{j}})^n|0\rangle &= \langle 0|(\mathrm{a}_{\boldsymbol{i}})^{n-1}\left([\mathrm{a}_{\boldsymbol{i}},(\mathrm{a}^\dagger_{\boldsymbol{j}})^n] + (\mathrm{a}^\dagger_{\boldsymbol{j}})^n\mathrm{a}_{\boldsymbol{i}}\right)|0\rangle = \langle 0|(\mathrm{a}_{\boldsymbol{j}})^{n-1}[\mathrm{a}_{\boldsymbol{j}},(\mathrm{a}^\dagger_{\boldsymbol{i}})^n]|0\rangle \\ &= \delta_{\boldsymbol{ji}}n\langle 0|(\mathrm{a}_{\boldsymbol{j}})^{n-1}(\mathrm{a}^\dagger_{\boldsymbol{i}})^{n-1}|0\rangle = \delta_{\boldsymbol{ji}}n(n-1)\langle 0|(\mathrm{a}_{\boldsymbol{j}})^{n-2}(\mathrm{a}^\dagger_{\boldsymbol{i}})^{n-2}|0\rangle ... \end{aligned} \tag{132}$$

Going further with the recurrence we get

$$\langle n_{\boldsymbol{i}}|n_{\boldsymbol{j}}\rangle \sim \delta_{\boldsymbol{ij}}n!\langle 0|0\rangle. \tag{133}$$

Now we can give a normalized definition of bras and kets:

$$\begin{aligned} |n_{\boldsymbol{j}}\rangle &= \frac{1}{\sqrt{n!}}(\mathrm{a}^\dagger_{\boldsymbol{j}})^n|0\rangle, \quad \boldsymbol{j}=1,2,3, \\ \langle n_{\boldsymbol{j}}| &= \frac{1}{\sqrt{n!}}\langle 0|\,(\mathrm{a}_{\boldsymbol{j}})^n\,, \quad \boldsymbol{j}=1,2,3. \end{aligned} \tag{134}$$

This means that for this representation the action of raising and lowering operators is defined as follows:

$$\mathrm{a}^\dagger_{\boldsymbol{j}}|n_{\boldsymbol{j}}\rangle = \sqrt{n+1}|n_{\boldsymbol{j}}+1\rangle, \tag{135}$$

$$\mathrm{a}_{\boldsymbol{j}}|n_{\boldsymbol{j}}\rangle = \sqrt{n}|n_{\boldsymbol{j}}-1\rangle, \tag{136}$$

$$\langle n_{\boldsymbol{j}}|\mathrm{a}_{\boldsymbol{j}} = \sqrt{n+1}\langle n_{\boldsymbol{j}}+1|, \tag{137}$$

$$\langle n_{\boldsymbol{j}}|\mathrm{a}^\dagger_{\boldsymbol{j}} = \sqrt{n}\langle n_{\boldsymbol{j}}-1|. \tag{138}$$

We also define the number operators for these three polarization degrees of freedom

$$\mathrm{a}_{\boldsymbol{j}}^{\dagger}\mathrm{a}_{\boldsymbol{j}}|n_{\boldsymbol{j}}\rangle \;=\; n_{\boldsymbol{j}}|n_{\boldsymbol{j}}\rangle, \qquad \boldsymbol{j}=1,2,3. \tag{139}$$

For further analysis, we will also need the following commutation relations

$$\begin{aligned} [\mathrm{a}_{\boldsymbol{j}}, \mathrm{a}_{\boldsymbol{j}}^{\dagger}a_{\boldsymbol{j}}] &= \mathrm{a}_{\boldsymbol{j}}, \qquad \boldsymbol{j}=1,2,3, \\ [\mathrm{a}_{\boldsymbol{j}}^{\dagger}, \mathrm{a}_{\boldsymbol{j}}^{\dagger}a_{\boldsymbol{j}}] &= -\mathrm{a}_{\boldsymbol{j}}^{\dagger}, \qquad \boldsymbol{j}=1,2,3. \end{aligned} \tag{140}$$

Using the definition of the lowering operator (111), with the definition of the vacuum state (130), for the q representation of vacuum, we can write a differential equation

$$\frac{1}{\sqrt{2}}\left(q_{\boldsymbol{j}}+\frac{\partial}{\partial q_{\boldsymbol{j}}}\right)\Omega_0(q_{\boldsymbol{j}}) \;=\; 0, \qquad \boldsymbol{j}=1,2,3, \tag{141}$$

with a simple solution

$$\Omega_0(q_{\boldsymbol{j}}) \;=\; A\exp\left(-\frac{q_{\boldsymbol{j}}^2}{2}\right), \qquad \boldsymbol{j}=1,2,3. \tag{142}$$

Here A is the normalization constant

$$A \;=\; \left(\frac{1}{\pi}\right)^{\frac{1}{4}}. \tag{143}$$

The whole construction for these three polarization degrees of freedom is evident. Now, formally without any confusion we can say that operators that raise the total energy create particles, so can be called creation operators. Also those that lower the total energy should annihilate the vacuum. Furthermore, the states have positive norms and the vacuum state in the canonical q representation is proper, i.e. satisfies the requirement of going to zero at infinity, making it possible to normalize the function.

3.3 Construction for time-like photons, where lowering energy operators annihilate vacuum

The problem arises with the $\boldsymbol{a}=0$ degree. To see this, first let us take a look at the time-like part of the Hamiltonian

$$\begin{aligned} H_0|n_1,n_2,n_3,n_0\rangle \;=\; \left(-\frac{p_0p^0}{2}-\frac{q_0q^0}{2}\right)|n_1,n_2,n_3,n_0\rangle \;&=\; \left(-a_0^{\dagger}a_0+\frac{1}{2}\right)|n_1,n_2,n_3,n_0\rangle \\ &=\; \left(-a_0a_0^{\dagger}-\frac{1}{2}\right)|n_1,n_2,n_3,n_0\rangle. \end{aligned} \tag{144}$$

When we consider the energy the question for the 0 polarization degree of freedom is: do the lowering energy operators annihilate particles and raising energy operators create ones or maybe do the lowering energy operators create particles and raising energy operators annihilate them?

First let us assume that the lowering operator annihilates the vacuum, i.e.

$$\mathrm{a}_0|0\rangle \;=\; 0. \tag{145}$$

Then the state of n_0 excitations is proportional to n raising operators acting on ground state, so that

$$|n_0\rangle \;\sim\; \mathrm{a}_0^{\dagger n}|0\rangle \tag{146}$$

and the scalar product reads

$$\begin{aligned} \langle n_0|n_0\rangle \;\sim\; \langle 0|(\mathrm{a}_0)^n\mathrm{a}_0^{\dagger n}|0\rangle \;&=\; \langle 0|(\mathrm{a}_0)^{n-1}\left([\mathrm{a}_0,\mathrm{a}_0^{\dagger n}]+\mathrm{a}_0^{\dagger n}\mathrm{a}_0\right)|0\rangle \;=\; \langle 0|(\mathrm{a}_0)^{n-1}[\mathrm{a}_0,\mathrm{a}_0^{\dagger n}]|0\rangle \\ &=\; n\langle 0|(-)(\mathrm{a}_0)^{n-1}\mathrm{a}_0^{\dagger n-1}|0\rangle \;=\; (-)^2n(n-1)\langle 0|(\mathrm{a}_0)^{n-2}\mathrm{a}_0^{\dagger n-2}|0\rangle ... \end{aligned} \tag{147}$$

Going further with the recurrence we get

$$\langle n_0|n_0\rangle \;\sim\; (-)^n n!\langle 0|0\rangle. \tag{148}$$

It looks like the states corresponding to odd values of n_0 give a negative norm. Normalization to 1 becomes a problem now and there is no elegant way to do it, maybe:

$$\begin{aligned} |n_0\rangle &= \frac{1}{\sqrt{n!}}(i\mathrm{a}_0^\dagger)^n|0\rangle, \\ \langle n_0| &= \frac{1}{\sqrt{n!}}\langle 0|\,(i\mathrm{a}_0)^n\,. \end{aligned} \tag{149}$$

In this case saving the positivity of the scalar product has an effect on the hermitian conjugation operation. This means that for this kind of representation the action of creation and annihilation operators would be defined as:

$$\mathrm{a}_0^\dagger|n_0\rangle = -i\sqrt{n+1}|n_0+1\rangle, \tag{150}$$
$$\mathrm{a}_0|n_0\rangle = -i\sqrt{n}|n_0-1\rangle, \tag{151}$$
$$\langle n_0|\mathrm{a}_0 = -i\sqrt{n+1}\langle n_0+1|, \tag{152}$$
$$\langle n_0|\mathrm{a}_0^\dagger = -i\sqrt{n}\langle n_0-1|. \tag{153}$$

Or we could follow Gupta [30] and leave the metric indefinite, i.e.

$$\begin{aligned} |n_0\rangle &= \frac{1}{\sqrt{n!}}(\mathrm{a}_0^\dagger)^n|0\rangle, \\ \langle n_0| &= \frac{1}{\sqrt{n!}}\langle 0|\,(\mathrm{a}_0)^n\,, \end{aligned} \tag{154}$$

defining the action of creation and annihilation operators as:

$$\mathrm{a}_0^\dagger|n_0\rangle = \sqrt{n+1}|n_0+1\rangle, \tag{155}$$
$$\mathrm{a}_0|n_0\rangle = -\sqrt{n}|n_0-1\rangle, \tag{156}$$
$$\langle n_0|\mathrm{a}_0 = \sqrt{n+1}\langle n_0+1|, \tag{157}$$
$$\langle n_0|\mathrm{a}_0^\dagger = -\sqrt{n}\langle n_0-1|. \tag{158}$$

But even then another problem arises, i.e. using the q representation of vacuum, we get a differential equation

$$\frac{1}{\sqrt{2}}\left(q_0 - \frac{\partial}{\partial q_0}\right)\Omega_0(q_0) \;=\; 0, \tag{159}$$

with a solution that is divergent at infinity

$$\Omega_0(q_0) \;=\; A\exp\left(\frac{q_0^2}{2}\right). \tag{160}$$

All these conclusions may suggest that another point of view on the 0 polarization degree is needed.

3.4 Construction for time-like photons, where raising energy operators annihilate vacuum

Now let us assume that the raising operator annihilates the vacuum, which means that the energy spectrum is bounded from the top and to raise the total energy level we need to annihilate a particle, i.e.

$$\mathrm{a}_0^\dagger|0\rangle \;=\; 0. \tag{161}$$

Such a construction has lots of advantages which will be shown in this section. Now the state of n_0 excitations is proportional to n lowering energy operators acting on the ground state, so that

$$|n_0\rangle \;\sim\; \mathrm{a}_0^n|0\rangle. \tag{162}$$

This means that creation of new particles is lowering the total energy. The scalar product now can be written as

$$\begin{aligned}\langle n_0|n_0\rangle \;\sim\; \langle 0|\mathrm{a}_0^{\dagger n}(\mathrm{a}_0)^n|0\rangle \;&=\; \langle 0|(\mathrm{a}_0^\dagger)^{n-1}\left([\mathrm{a}_0^\dagger,\mathrm{a}_0^n]+\mathrm{a}_0^n\mathrm{a}_0^\dagger\right)|0\rangle \;=\; \langle 0|(\mathrm{a}_0^\dagger)^{n-1}[\mathrm{a}_0^\dagger,\mathrm{a}_0^n]|0\rangle\\ &=\; n\langle 0|(\mathrm{a}_0^\dagger)^{n-1}\mathrm{a}_0^{n-1}|0\rangle \;=\; n(n-1)\langle 0|(\mathrm{a}_0^\dagger)^{n-2}\mathrm{a}_0^{n-2}|0\rangle ... \end{aligned} \tag{163}$$

Going further with the recurrence we get

$$\langle n_0|n_0\rangle \;\sim\; n!\langle 0|0\rangle. \tag{164}$$

We find that giving a normalized definition of bras and kets is not a problem now:

$$\begin{aligned}|n_0\rangle &= \frac{1}{\sqrt{n!}}\mathrm{a}_0^n|0\rangle,\\ \langle n_0| &= \frac{1}{\sqrt{n!}}\langle 0|\mathrm{a}_0^{\dagger n}.\end{aligned} \tag{165}$$

Furthermore, this means that for this representation the raising and lowering energy operators are defined as:

$$\mathrm{a}_0|n_0\rangle = \sqrt{n+1}|n_0+1\rangle, \tag{166}$$

$$\mathrm{a}_0^\dagger|n_0\rangle = \sqrt{n}|n_0-1\rangle, \tag{167}$$

$$\langle n_0|\mathrm{a}_0^\dagger = \sqrt{n+1}\langle n_0+1|, \tag{168}$$

$$\langle n_0|\mathrm{a}_0 = \sqrt{n}\langle n_0-1|. \tag{169}$$

It turns out that in this case we do not have to choose between positivity of the scalar product and the hermitian conjugate operation. The number operator for the time-like polarization should be defined carefully:

$$\mathrm{a}_0\mathrm{a}_0^\dagger|n_0\rangle = n_0|n_0\rangle. \tag{170}$$

Of course, we can use an operator in the “standard way” but then its eigenvalue would be n_0+1

$$\mathrm{a}_0^\dagger\mathrm{a}_0|n_0\rangle = (n_0+1)|n_0\rangle. \tag{171}$$

A construction of vacuum state is not problematic either. Using the q representation we get a differential equation

$$\frac{1}{\sqrt{2}}\left(q_0+\frac{\partial}{\partial q_0}\right)\Omega_0(q_0) \;=\; 0, \tag{172}$$

with the solution

$$\Omega_0(q_0) \;=\; A\exp\left(\frac{-q_0^2}{2}\right). \tag{173}$$

Summarizing all the above, from now on we will use $a_0^\dagger$ for an operator that annihilates vacuum and a_0 for an operator that creates time-like polarization states.

3.5 More properties of the four-dimensional oscillator algebra

We can split the Hamiltonian in two: for degrees 1 and 2, and for 0 and 3, such that

$$H|n_1,n_2,n_3,n_0\rangle \;=\; \left(H^{12}+H^{03}\right)|n_1,n_2,n_3,n_0\rangle, \tag{174}$$

$$\begin{aligned}H^{12}|n_1,n_2,n_3,n_0\rangle &= -\frac{p_1p^1+q_1q^1+p_2p^2+q_2q^2}{2}|n_1,n_2,n_3,n_0\rangle\\ &= \left(a_1^\dagger a_1+a_2^\dagger a_2+1\right)|n_1,n_2,n_3,n_0\rangle \;=\; (n_1+n_2+1)\,|n_1,n_2,n_3,n_0\rangle,\end{aligned} \tag{175}$$

$$
\begin{aligned}
H^{03}|n_1,n_2,n_3,n_0\rangle &= -\frac{p_0p^0+q_0q^0+p_3p^3+q_3q^3}{2}|n_1,n_2,n_3,n_0\rangle \\
&= \left(-a_0^\dagger a_0 + a_3^\dagger a_3 + 1\right)|n_1,n_2,n_3,n_0\rangle \\
&= \left(-a_0 a_0^\dagger + a_3^\dagger a_3\right)|n_1,n_2,n_3,n_0\rangle \; = \; (-n_0+n_3)\,|n_1,n_2,n_3,n_0\rangle. \qquad (176)
\end{aligned}
$$

From this we see that, for such a definition of a_0, the energy of the ground state comes only from the transverse polarization degrees of freedom.

The completeness relation for such four-dimensional oscillator holds

$$
\begin{aligned}
&\sum_{n_1,n_2,n_3,n_0=0}^{\infty} |n_1,n_2,n_3,n_0\rangle\langle n_1,n_2,n_3,n_0| \\
&= \sum_{n_1=0}^{\infty}|n_1\rangle\langle n_1| \otimes \sum_{n_2=0}^{\infty}|n_2\rangle\langle n_2| \otimes \sum_{n_3=0}^{\infty}|n_3\rangle\langle n_3| \otimes \sum_{n_0=0}^{\infty}|n_0\rangle\langle n_0| \; = \; 1_4. \qquad (177)
\end{aligned}
$$

3.6 Conclusions and results

This chapter contains new results. A construction of the four-dimensional polarization space coming from a definition of the covariant Hamiltonian (108) is presented here. Further analysis of the formalism is performed, especially regarding the interpretation of the ladder operators for the time-like degree of freedom a_0. Strong arguments are given in favor of an interpretation in which the operator annihilating vacuum is a raising energy operator. Such an interpretation gives a non divergent vacuum representation and positive scalar products. These results are in agreement with the four-dimensional quantization of the potential operator in Czachor and Naudts [12], and Czachor and Wrzask [13]. One thing should be mentioned regarding the choice of notation a_0 for the creation operator of the time-like polarization degree of freedom, i.e. without a dagger. This differs from the notation used in the mentioned papers, but follows quite naturally from the definition of the covariant Hamiltonian (108). Furthermore, such a choice of notation allows us to write formulas in a more compact and covariant looking forms.

4 Four-dimensional oscillator reducible representation algebra

In previous two chapters the reducible representations as well as the four-dimensional oscillator algebra were introduced. This chapter in a way combines the two previous ones in a single formalism. First, a construction of the four-dimensional oscillator for the reducible representation is given in 4.1 and 4.2. Next, the potential operator for the theory is introduced in 4.3. Then a question comes up: how does such a theory correspond to Maxwell electromagnetism theory. It turns out that states that reproduce standard Maxwell electrodynamics exist. This will be discussed in section 4.4, and an explicit form of Ψ_{EM} states is a new result. In 4.5 more properties of the potential operator are shown. This includes the covariant commutator taken in arbitrary space-time points coming from the covariant structure of the ladder operators. Some strong arguments are given here in favor of the reducible representation of the theory. The electromagnetic field tensor is introduced in 4.6 and coherent states in 4.7. At the end of this chapter, in 4.8, a coherent state structure of Ψ_{EM} vectors is discussed.

4.1 Hamiltonian

The main goal of this section is to combine the reducible representation with the covariant construction of the four polarization degrees of freedom into one model. The four-dimensional $N = 1$ (or 1–oscillator) representation of CCR acts in the Hilbert space $\mathcal{H}(1)$ spanned by kets of the form

$$|\boldsymbol{k}, n_1, n_2, n_3, n_0\rangle = |\boldsymbol{k}\rangle \otimes \frac{(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}(a_3^\dagger)^{n_3}(a_0)^{n_0}}{\sqrt{n_0!n_1!n_2!n_3!}}|0,0,0,0\rangle \tag{178}$$

or

$$|\boldsymbol{k}, n_-, n_+, n_3, n_0\rangle = |\boldsymbol{k}\rangle \otimes \frac{(a_-^\dagger)^{n_1}(a_+^\dagger)^{n_2}(a_3^\dagger)^{n_3}(a_0)^{n_0}}{\sqrt{n_0!n_-!n_+!n_3!}}|0,0,0,0\rangle. \tag{179}$$

This 1-oscillator representation has four dimensions of polarization. a_1, a_2, a_3, a_0 satisfy the commutation relations typical of irreducible representations of CCR (113). In (178) $a_1^\dagger$, $a_2^\dagger$ stand for creation operators for linear polarized photons in x and y directions, and in (179) $a_+^\dagger$, $a_-^\dagger$ stand for circular polarization photons. a_0 and $a_3^\dagger$ are both creation operators for time-like and longitudinal photons, respectively. For the reducible representation we define the ladder operators

$$a_{\boldsymbol{a}}(\boldsymbol{k}, 1) = |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes a_{\boldsymbol{a}}. \tag{180}$$

Then the following CCR algebra holds:

$$[a_{\boldsymbol{a}}(\boldsymbol{k},1), a_{\boldsymbol{b}}(\boldsymbol{k}',1)^\dagger] = -g_{\boldsymbol{ab}}\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}')|\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes 1_4 = -g_{\boldsymbol{ab}}\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}')I(\boldsymbol{k},1). \tag{181}$$

This representation is reducible, since the right-hand side of the commutator (181) is an operator valued distribution $I(\boldsymbol{k}, 1) = |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes 1_4$ belonging to the center of the algebra, i.e.

$$[a_{\boldsymbol{a}}(\boldsymbol{k},1), I(\boldsymbol{k}',1)] = [a_{\boldsymbol{a}}(\boldsymbol{k},1)^\dagger, I(\boldsymbol{k}',1)] = 0. \tag{182}$$

In order to construct this four-dimensional oscillator space let us introduce a covariant Hamiltonian in the reducible representation

$$H(1) = \int d\Gamma(\boldsymbol{k})\ |\boldsymbol{k}|\ |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes \left(-\frac{p_{\boldsymbol{a}}p^{\boldsymbol{a}}}{2} - \frac{q_{\boldsymbol{a}}q^{\boldsymbol{a}}}{2}\right), \quad \boldsymbol{a} = 0,1,2,3. \tag{183}$$

Hamiltonian (183) can be also written in terms of operators $a_{\boldsymbol{a}}^\dagger$ and $a_{\boldsymbol{a}}$ in the form

$$H(1) = \int d\Gamma(\boldsymbol{k})\ |\boldsymbol{k}|\ |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes \left(-a_{\boldsymbol{a}}^\dagger a^{\boldsymbol{a}} + 2\right). \tag{184}$$

Now the following commutation relations hold for the reducible representation of ladder operators

$$[H(1), a_{\boldsymbol{a}}(\boldsymbol{k},1)] = -|\boldsymbol{k}|\ a_{\boldsymbol{a}}(\boldsymbol{k},1), \tag{185}$$
$$[H(1), a_{\boldsymbol{a}}(\boldsymbol{k},1)^\dagger] = |\boldsymbol{k}|\ a_{\boldsymbol{a}}(\boldsymbol{k},1)^\dagger. \tag{186}$$

Moreover, within the whole frequency spectrum of the ladder operators we get

$$[H(1), a_{\boldsymbol{a}}(1)] \;=\; -\int d\Gamma(\boldsymbol{k})\ |\boldsymbol{k}|\ a_{\boldsymbol{a}}(\boldsymbol{k},1) \;=\; -\,\Omega\otimes a_{\boldsymbol{a}}, \tag{187}$$

$$[H(1), a_{\boldsymbol{a}}(1)^\dagger] \;=\; \int d\Gamma(\boldsymbol{k})\ |\boldsymbol{k}|\ a_{\boldsymbol{a}}(\boldsymbol{k},1)^\dagger \;=\; \Omega\otimes a^\dagger_{\boldsymbol{a}}, \tag{188}$$

where spectral decomposition of the frequency operator (43) has been employed. Let us denote by $E(1)$ an eigenvalue of the covariant Hamiltonian operator (183),

$$H(1)|\boldsymbol{k},n_1,n_2,n_3,n_0\rangle \;=\; E(1)|\boldsymbol{k},n_1,n_2,n_3,n_0\rangle \;=\; |\boldsymbol{k}|\ (n_1+n_2+n_3-n_0+1)|\boldsymbol{k},n_1,n_2,n_3,n_0\rangle. \tag{189}$$

Now it can be shown that indeed $a_{\boldsymbol{a}}(1)$ lowers and $a^\dagger_{\boldsymbol{a}}(1)$ raises the total energy by $|\boldsymbol{k}|$, i.e.

$$\begin{aligned} H(1)a_{\boldsymbol{a}}(1)|\boldsymbol{k},n_1,n_2,n_3,n_0\rangle &= [H(1),a_{\boldsymbol{a}}(1)]|\boldsymbol{k},n_1,n_2,n_3,n_0\rangle + a_{\boldsymbol{a}}(1)H(1)|\boldsymbol{k},n_1,n_2,n_3,n_0\rangle \\ &= -\int d\Gamma(\boldsymbol{k}')\ |\boldsymbol{k}'|\ a_{\boldsymbol{a}}(\boldsymbol{k}',1)|\boldsymbol{k},n_1,n_2,n_3,n_0\rangle + a_{\boldsymbol{a}}(1)E(1)|\boldsymbol{k},n_1,n_2,n_3,n_0\rangle \\ &= (E(1)-|\boldsymbol{k}|)a_{\boldsymbol{a}}(1)|\boldsymbol{k},n_1,n_2,n_3,n_0\rangle, \end{aligned} \tag{190}$$

$$\begin{aligned} H(1)a_{\boldsymbol{a}}(1)^\dagger|\boldsymbol{k},n_1,n_2,n_3,n_0\rangle &= [H(1),a_{\boldsymbol{a}}(1)^\dagger]|\boldsymbol{k},n_1,n_2,n_3,n_0\rangle + a_{\boldsymbol{a}}(1)^\dagger H(1)|\boldsymbol{k},n_1,n_2,n_3,n_0\rangle \\ &= \int d\Gamma(\boldsymbol{k}')\ |\boldsymbol{k}'|\ a_{\boldsymbol{a}}(\boldsymbol{k}',1)^\dagger|\boldsymbol{k},n_1,n_2,n_3,n_0\rangle + a_{\boldsymbol{a}}(1)^\dagger E(1)|\boldsymbol{k},n_1,n_2,n_3,n_0\rangle \\ &= (E(1)+|\boldsymbol{k}|)a_{\boldsymbol{a}}(1)^\dagger|\boldsymbol{k},n_1,n_2,n_3,n_0\rangle. \end{aligned} \tag{191}$$

Going further, the Hilbert space for any N-oscillator in this representation reads,

$$\mathcal{H}(N) \;=\; \underbrace{\mathcal{H}(1)\otimes\ldots\otimes\mathcal{H}(1)}_{N} \;=\; \mathcal{H}(1)^{\otimes N}, \tag{192}$$

so that the $\mathcal{H}(N)$ Hilbert space is spanned by kets of the form

$$|\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N,n_1^1,\ldots,n_0^N\rangle \;=\; |\boldsymbol{k}_1,n_1^1,n_2^1,n_3^1,n_0^1\rangle\otimes\cdots\otimes|\boldsymbol{k}_N,n_1^N,n_2^N,n_3^N,n_0^N\rangle. \tag{193}$$

Recalling the definition of the number operator (89), we may construct the Hamiltonian for the N-oscillator representation as follows

$$\begin{aligned} H(N) &= \int d\Gamma(\boldsymbol{k})\ |\boldsymbol{k}|\ \sum_{n=1}^{N}\big(|\boldsymbol{k}\rangle\langle\boldsymbol{k}|\otimes\big(-a^\dagger_{\boldsymbol{a}}a^{\boldsymbol{a}}+2\big)\big)^{(n)} \\ &= \sum_{n=1}^{N}\left(\int d\Gamma(\boldsymbol{k})\ |\boldsymbol{k}|\ |\boldsymbol{k}\rangle\langle\boldsymbol{k}|\otimes\big(-a^\dagger_{\boldsymbol{a}}a^{\boldsymbol{a}}+2\big)\right)^{(n)} \;=\; \sum_{n=1}^{N}H(1)^{(n)}. \end{aligned} \tag{194}$$

In the next section it will be shown that this definition of the Hamiltonian makes the vacuum energy finite also for an arbitrary N-oscillator representation.

4.2 Vacuum in four-dimensional polarization reducible representations

The subspace of vacuum states is spanned by vectors of the form

$$|\boldsymbol{k}_1,0,0,0,0\rangle\otimes\cdots\otimes|\boldsymbol{k}_N,0,0,0,0\rangle. \tag{195}$$

Vacuum in this representation is any state annihilated by all annihilation operators. Let us recall that a_0 is in this model a creation operator. Therefore, in $N=1$ oscillator representation we may write

$$|O(1)\rangle \;=\; \int d\Gamma(\boldsymbol{k})O(\boldsymbol{k})|\boldsymbol{k},0,0,0,0\rangle. \tag{196}$$

In such a definition there can be a place for a single oscillator field $O(\boldsymbol{k})$, where:

$$\int d\Gamma(\boldsymbol{k})|O(\boldsymbol{k})|^2 \;=\; \int d\Gamma(\boldsymbol{k})Z(\boldsymbol{k}) \;=\; 1. \tag{197}$$

$|O(\boldsymbol{k})|^2 = Z(\boldsymbol{k})$ function may be understood as the probability that a given oscillator has momentum $\boldsymbol{k}$. An extension to N-oscillator representation can be made by

$$|O(N)\rangle \;=\; \underbrace{|O(1)\rangle \otimes ... \otimes |O(1)\rangle}_{N} \;=\; |O(1)\rangle^{\otimes N}. \tag{198}$$

From (197) we get the normalization condition

$$\langle O(N)|O(N)\rangle \;=\; \langle O(1)|O(1)\rangle^{N} \;=\; 1. \tag{199}$$

Recalling (176), where it is shown that the energy of vacuum comes only from the two transverse polarization degrees of freedom, we calculate

$$\begin{aligned}
H(1)|O(1)\rangle \;&=\; \int d\Gamma(\boldsymbol{k})\ |\boldsymbol{k}|\ |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes \left(-a_{\boldsymbol{a}}^{\dagger}a^{\boldsymbol{a}} + 2\right) \int d\Gamma(\boldsymbol{k}')O(\boldsymbol{k}')|\boldsymbol{k}',0,0,0,0\rangle \\
&=\; \int d\Gamma(\boldsymbol{k}) \int d\Gamma(\boldsymbol{k}')\ |\boldsymbol{k}|\ \delta_{\Gamma}(\boldsymbol{k},\boldsymbol{k}')O(\boldsymbol{k}')|\boldsymbol{k},0,0,0,0\rangle \\
&=\; \int d\Gamma(\boldsymbol{k})\ |\boldsymbol{k}|\ O(\boldsymbol{k})|\boldsymbol{k},0,0,0,0\rangle.
\end{aligned} \tag{200}$$

Let us also take into account the definition (194) of the Hamiltonian in N-oscillator representation, and write

$$H(N)|O(N)\rangle \;=\; \sum_{n=1}^{N} H(1)^{(n)}|O(N)\rangle \;=\; \sum_{m=1}^{N} |O(1)\rangle^{\otimes(m-1)} \otimes H(1)|O(1)\rangle \otimes |O(1)\rangle^{\otimes(N-m)}. \tag{201}$$

It must be stressed that in this case the expectation value of the energy of vacuum does not depend on N, i.e.

$$\langle O(N)|H(N)|O(N)\rangle \;=\; \langle O(1)|O(1)\rangle^{N-1}\langle O(1)|H(1)|O(1)\rangle \;=\; \int d\Gamma(\boldsymbol{k})\ |\boldsymbol{k}|\ Z(\boldsymbol{k}). \tag{202}$$

This means that the energy of vacuum is not zero and depends only on the vacuum probability density $Z(\boldsymbol{k})$. Furthermore, for the vacuum energy to be finite we must demand

$$\int d\Gamma(\boldsymbol{k})\ |\boldsymbol{k}|\ Z(\boldsymbol{k}) < \infty. \tag{203}$$

It is possible to find such a function, for example Gaussians fulfill condition (203). Therefore, finiteness of average vacuum energy is guaranteed by a proper choice of the vacuum probability density function and, frankly, does not require the N parameter at all. Furthermore, this analysis shows that N may be a finite number.

4.3 The potential operator

Four-dimensional quantization, as presented here, was formulated by Czachor and Naudts in [12] and further investigated by Czachor and Wrzask in [13], where in the place of the annihilation operator of the Gupta-Bleuler potential, for the time-like degree of freedom, stands a creation operator and vice-versa. Let us start by presenting this potential operator in the reducible representation for $N = 1$:

$$\begin{aligned}
A_a(x,1) \;&=\; i\int d\Gamma(\boldsymbol{k})g_a{}^{\boldsymbol{a}}(\boldsymbol{k})a_{\boldsymbol{a}}(\boldsymbol{k},1)e^{-ik\cdot x} + \text{H.c.} \\
&=\; i\int d\Gamma(\boldsymbol{k})\left(g_a{}^{1}(\boldsymbol{k})a_1(\boldsymbol{k},1) + g_a{}^{2}(\boldsymbol{k})a_2(\boldsymbol{k},1) + g_a{}^{3}(\boldsymbol{k})a_3(\boldsymbol{k},1) + g_a{}^{0}(\boldsymbol{k})a_0(\boldsymbol{k},1)\right)e^{-ik\cdot x} + \text{H.c.} \\
&=\; i\int d\Gamma(\boldsymbol{k})\left(-x_a(\boldsymbol{k})a_1(\boldsymbol{k},1) - y_a(\boldsymbol{k})a_2(\boldsymbol{k},1) - z_a(\boldsymbol{k})a_3(\boldsymbol{k},1) + t_a(\boldsymbol{k})a_0(\boldsymbol{k},1)\right)e^{-ik\cdot x} + \text{H.c}
\end{aligned} \tag{204}$$

Here not just two operators corresponding to the polarization degrees of freedom, but four types are introduced. Let us stress that a_0 accompanying $t_a(\boldsymbol{k})$ is indeed a lowering total energy operator that is a creation operator like in [12]. It is denoted differently here, i.e. without a dagger, as a result of the analysis made in the previous chapter, and for notational convenience, i.e. the possibility of writing collective formulas. In (204) $x_a(\boldsymbol{k})$, $y_a(\boldsymbol{k})$, $z_a(\boldsymbol{k})$, $t_a(\boldsymbol{k})$ is a field of Minkowski tetrads such that

$$k_a(\boldsymbol{k})x^a(\boldsymbol{k}) \;=\; k_a(\boldsymbol{k})y^a(\boldsymbol{k}) \;=\; 0, \qquad k_a(\boldsymbol{k})z^a(\boldsymbol{k}) \;=\; k_a(\boldsymbol{k})t^a(\boldsymbol{k}) \;=\; \frac{1}{\sqrt{2}}. \tag{205}$$

Now let us take a closer look at the part

$$\begin{aligned}
g_a{}^{\boldsymbol{a}}(\boldsymbol{k})a_{\boldsymbol{a}}(\boldsymbol{k},1) &= g_a{}^{\boldsymbol{a}'}(\boldsymbol{k})g_{\boldsymbol{a}'}{}^{\boldsymbol{a}}a_{\boldsymbol{a}}(\boldsymbol{k},1) = g_a{}^{\boldsymbol{a}'}(\boldsymbol{k})\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1\\ 0 & 1 & -i & 0\\ 0 & 1 & i & 0\\ 1 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix} a_0(\boldsymbol{k},1)\\ a_1(\boldsymbol{k},1)\\ a_2(\boldsymbol{k},1)\\ a_3(\boldsymbol{k},1)\end{pmatrix}\\
&= g_a{}^{\boldsymbol{a}'}(\boldsymbol{k})\frac{1}{\sqrt{2}}\begin{pmatrix} a_0(\boldsymbol{k},1)+a_3(\boldsymbol{k},1)\\ a_1(\boldsymbol{k},1)-ia_2(\boldsymbol{k},1)\\ a_1(\boldsymbol{k},1)+ia_2(\boldsymbol{k},1)\\ a_0(\boldsymbol{k},1)-a_3(\boldsymbol{k},1)\end{pmatrix} = g_a{}^{\boldsymbol{a}'}(\boldsymbol{k})a_{\boldsymbol{a}'}(\boldsymbol{k},1)\\
&= -\bar{m}_a(\boldsymbol{k})a_{01'}(\boldsymbol{k},1) - m_a(\boldsymbol{k})a_{10'}(\boldsymbol{k},1) + k_a(\boldsymbol{k})a_{00'}(\boldsymbol{k},1) + \omega_a(\boldsymbol{k})a_{11'}(\boldsymbol{k},1).
\end{aligned} \tag{206}$$

As we can see this can be rewritten also in terms of the null tetrad. Commutation relations for the new operators are

$$\begin{aligned}
[a_{\boldsymbol{a}'}(\boldsymbol{k},1), a_{\boldsymbol{b}'}(\boldsymbol{k},1)^\dagger] &= [g_{\boldsymbol{a}'}{}^{\boldsymbol{a}}a_{\boldsymbol{a}}(\boldsymbol{k},1), \bar{g}^{\boldsymbol{b}}{}_{\boldsymbol{b}'}a_{\boldsymbol{b}}(\boldsymbol{k},1)^\dagger] = -\,g_{\boldsymbol{a}'}{}^{\boldsymbol{a}}\bar{g}^{\boldsymbol{b}}{}_{\boldsymbol{b}'}g_{\boldsymbol{ab}} = -\,g_{\boldsymbol{a}'}{}^{\boldsymbol{a}}\bar{g}_{\boldsymbol{ab}'}\\
&= -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1\\ 0 & 1 & -i & 0\\ 0 & 1 & i & 0\\ 1 & 0 & 0 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1\\ 0 & -1 & -1 & 0\\ 0 & -i & i & 0\\ -1 & 0 & 0 & 1\end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -1 & 0 & 0 & 0\end{pmatrix}.
\end{aligned} \tag{207}$$

So, the electromagnetic four-potential operator can be also defined in another way, i.e.

$$\begin{aligned}
&A_a(x,1)\\
=\;& i\int d\Gamma(\boldsymbol{k})\left(-m_a(\boldsymbol{k})a_-(\boldsymbol{k},1) - \bar{m}_a(\boldsymbol{k})a_+(\boldsymbol{k},1) - z_a(\boldsymbol{k})a_3(\boldsymbol{k},1) + t_a(\boldsymbol{k})a_0(\boldsymbol{k},1)\right)e^{-ik\cdot x} + \text{H.c.}\\
=\;& i\int d\Gamma(\boldsymbol{k})\left(-m_a(\boldsymbol{k})a_-(\boldsymbol{k},1) - \bar{m}_a(\boldsymbol{k})a_+(\boldsymbol{k},1) + k_a(\boldsymbol{k})a_{00'}(\boldsymbol{k},1) + \omega_a(\boldsymbol{k})a_{11'}(\boldsymbol{k},1)\right)e^{-ik\cdot x} + \text{H.c.}
\end{aligned} \tag{208}$$

The null tetrad fulfills the transversality property

$$k_a(\boldsymbol{k})m^a(\boldsymbol{k}) \;=\; k_a(\boldsymbol{k})\bar{m}^a(\boldsymbol{k}) \;=\; k_a(\boldsymbol{k})k^a(\boldsymbol{k}) \;=\; 0, \tag{209}$$

but

$$k_a(\boldsymbol{k})\omega^a(\boldsymbol{k}) \;=\; 1. \tag{210}$$

In this sense, from (205) and (209), both definitions of the four-vector potential operator involve four polarization degrees of freedom (two transverse, one longitudinal, and one time-like). The first definition (204) involves photons of linear polarization in x and y directions, the second one (208) deals with photons of circular polarization, where

$$a_{10'}(\boldsymbol{k},1) \;=\; a_-(\boldsymbol{k},1) \;=\; \frac{1}{\sqrt{2}}\left(a_1(\boldsymbol{k},1) + i\; a_2(\boldsymbol{k},1)\right), \tag{211}$$

$$a_{01'}(\boldsymbol{k},1) \;=\; a_+(\boldsymbol{k},1) \;=\; \frac{1}{\sqrt{2}}\left(a_1(\boldsymbol{k},1) - i\; a_2(\boldsymbol{k},1)\right), \tag{212}$$

$$a_{00'}(\boldsymbol{k},1) \;=\; \frac{1}{\sqrt{2}}\left(a_0(\boldsymbol{k},1) + a_3(\boldsymbol{k},1)\right), \tag{213}$$

$$a_{11'}(\boldsymbol{k},1) \;=\; \frac{1}{\sqrt{2}}\left(a_0(\boldsymbol{k},1) - a_3(\boldsymbol{k},1)\right), \tag{214}$$

and

$$\bar{m}_a(\boldsymbol{k}) = \frac{x_a(\boldsymbol{k}) + iy_a(\boldsymbol{k})}{\sqrt{2}}, \qquad m_a(\boldsymbol{k}) = \frac{x_a(\boldsymbol{k}) - iy_a(\boldsymbol{k})}{\sqrt{2}}, \tag{215}$$

play the role of circular polarization vectors.

4.4 Vectors corresponding to Maxwell's theory

This section contains new results and the main goal here is to introduce state vectors corresponding to classical Maxwell's theory. We will start by presenting the four-divergence of the potential operator in $N = 1$ representation

$$\begin{aligned}
\partial^a A_a(x,1) &= \partial^a \left(i \int d\Gamma(\boldsymbol{k}) g_a{}^{\boldsymbol{a}}(\boldsymbol{k}) a_{\boldsymbol{a}}(\boldsymbol{k},1) e^{-ik\cdot x} + \text{H.c.} \right) \\
&= \int d\Gamma(\boldsymbol{k}) k^a(\boldsymbol{k}) g_a{}^{\boldsymbol{a}}(\boldsymbol{k}) a_{\boldsymbol{a}}(\boldsymbol{k},1) e^{-ik\cdot x} + \text{H.c.} \\
&= \int d\Gamma(\boldsymbol{k}) \left(-z_a(\boldsymbol{k}) k^a(\boldsymbol{k}) a_3(\boldsymbol{k},1) + t_a(\boldsymbol{k}) k^a(\boldsymbol{k}) a_0(\boldsymbol{k},1) \right) e^{-ik\cdot x} + \text{H.c.} \\
&= \frac{1}{\sqrt{2}} \int d\Gamma(\boldsymbol{k}) \left(a_0(\boldsymbol{k},1) - a_3(\boldsymbol{k},1) \right) e^{-ik\cdot x} + \text{H.c.}
\end{aligned} \tag{216}$$

As one can see this does not correspond to the Lorenz condition on the four-vector potential operator. The operator $a_0(\boldsymbol{k},1) - a_3(\boldsymbol{k},1)$ in Dürr's and Rudolph's paper [34] is called a "bad ghost". This is a very adequate name because it spoils the correspondence with classical electrodynamics. It will appear later in section (5.11) in (419) spoiling the Lorenz gauge invariance of electromagnetic field operator. (216) is Lorentz invariant due to the invariance of the operator $a_0(\boldsymbol{k},1) - a_3(\boldsymbol{k},1)$. This is shown later in section (5.12). This is a good news, because once we could eliminate in averages the bad ghosts, we can be sure that they will never "spook" us in any other reference frame.

Returning to our problem, it would be good for the theory to eliminate "bad ghosts" and this can be done by a weaker Lorenz condition, i.e. in averages

$$\langle \Psi_{EM}(1) | \partial^a A_a(x,1) | \Psi_{EM}(1) \rangle = 0. \tag{217}$$

We will try to impose this condition on the Hilbert space instead of on the operators. Here $\Psi_{EM}(1)$ are vectors that satisfy (217). From (217) it follows that

$$\langle \Psi_{EM}(1) | \int d\Gamma(\boldsymbol{k}) \left(a_0(\boldsymbol{k},1) - a_3(\boldsymbol{k},1) \right) | \Psi_{EM}(1) \rangle = 0. \tag{218}$$

Let us first assume that vectors $\Psi_{EM}(1)$ can be split into two parts and written as a tensor product of transverse degrees of freedom vectors denoted by $\Psi_{12}(1)$, and 0 and 3 degrees of freedom vectors denoted by $\Psi_{03}(1)$.

$$\begin{aligned}
|\Psi_{EM}(1)\rangle &= \sum_{n_0,n_1,n_2,n_3}^{\infty} \int d\Gamma(\boldsymbol{k})\ \Psi_{EM}(\boldsymbol{k},n_1,n_2,n_3,n_0) |\boldsymbol{k},n_1,n_2,n_3,n_0\rangle \\
&= \sum_{n_1,n_2}^{\infty} \int d\Gamma(\boldsymbol{k})\ \Psi_{12}(\boldsymbol{k},n_1,n_2) |\boldsymbol{k},n_1,n_2\rangle \sum_{n_0,n_3}^{\infty} \Psi_{03}(n_3,n_0) |n_3,n_0\rangle \\
&= |\Psi_{12}(1)\rangle \otimes |\Psi_{03}(1)\rangle.
\end{aligned} \tag{219}$$

Let us note that such a choice is not the most generalized one because $\Psi_{03}(n_3, n_0)$ could also depend on $\boldsymbol{k}$. Now (218) takes the form

$$\begin{aligned}
& \langle \Psi_{EM}(1)| \left(\int d\Gamma(\boldsymbol{k}) |\boldsymbol{k}\rangle\langle \boldsymbol{k}| \otimes (a_0 - a_3)_4 \right) \\
\times & \sum_{n_1,n_2}^{\infty} \int d\Gamma(\boldsymbol{k}')\ \Psi_{12}(\boldsymbol{k}', n_1, n_2)|\boldsymbol{k}', n_1, n_2\rangle \sum_{n_0,n_3}^{\infty} \Psi_{03}(n_3, n_0)|n_3, n_0\rangle \\
= & \langle \Psi_{EM}(1)| \left(I \otimes (a_0 - a_3)_4\right) \sum_{n_1,n_2}^{\infty} \int d\Gamma(\boldsymbol{k})\ \Psi_{12}(\boldsymbol{k}, n_1, n_2)|\boldsymbol{k}, n_1, n_2\rangle \sum_{n_0,n_3}^{\infty} \Psi_{03}(n_3, n_0)|n_3, n_0\rangle \\
= & \langle \Psi_{12}(1)| \otimes \langle \Psi_{03}(1)| \left(I \otimes 1_2 \otimes (a_0 - a_3)_2\right) |\Psi_{12}(1)\rangle \otimes |\Psi_{03}(1)\rangle \\
= & \langle \Psi_{12}(1)|I \otimes 1_2|\Psi_{12}(1)\rangle \times \langle \Psi_{03}(1)| \left(a_0 - a_3\right)_2 |\Psi_{03}(1)\rangle. \qquad (220)
\end{aligned}$$

The lower indices in ladder operators indicate four and two dimensional tensor product spaces. So the week Lorenz condition can be written just in terms of $\Psi_{03}(1)$ vectors

$$\langle \Psi_{03}(1)| \left(a_0 - a_3\right)_2 |\Psi_{03}(1)\rangle \quad = \quad 0. \qquad (221)$$

Now, from (221), a condition on the $\Psi_{03}(n_3, n_0)$ function follows

$$\sum_{n_0=0,n_3=0}^{\infty} \left(\sqrt{n_0+1}\bar{\Psi}_{03}(n_0+1, n_3)\Psi_{03}(n_0, n_3) - \sqrt{n_3+1}\bar{\Psi}_{03}(n_0, n_3)\Psi_{03}(n_0, n_3+1) \right) \quad = \quad 0. \qquad (222)$$

This is derived step by step in appendix (D.1). It is possible to find such a normalized function, for example

$$\Psi_{03}(n_0, n_3) \quad = \quad \frac{e^{-1}}{\sqrt{n_0! n_3!}}. \qquad (223)$$

Here we can see a coherent-like structure of (223). This will be investigated further in section 4.8. Therefore, the $\Psi_{EM}(1)$ vectors have the form

$$|\Psi_{EM}(1)\rangle \quad = \quad \sum_{n_0,n_1,n_2,n_3}^{\infty} \frac{e^{-1}}{\sqrt{n_0! n_3!}} \int d\Gamma(\boldsymbol{k})\ \Psi_{12}(\boldsymbol{k}, n_1, n_2)|\boldsymbol{k}, n_1, n_2, n_3, n_0\rangle. \qquad (224)$$

It should be stressed that this condition is not the usual Gupta-Bleuler condition such that

$$\left(a_0(\boldsymbol{k}, 1) - a_3(\boldsymbol{k}, 1)\right) |\Psi(1)\rangle \quad = \quad 0, \qquad (225)$$

due to two aspects: a different definition of a_0 ladder operator, and because it holds not on the vector states but on the inner products.

Furthermore, it can be shown that for $\Psi_{EM}(1)$ vectors the number of time-like photons is equal to the number of longitudinal ones, i.e.

$$\langle \Psi_{EM}(1)| \left(n_0(\boldsymbol{k}, 1) - n_3(\boldsymbol{k}, 1)\right) |\Psi_{EM}(1)\rangle \quad = \quad 0. \qquad (226)$$

Using the definition of the number operator for $N = 1$ reducible representations (59), and having in mind the definition for the time-like ladder operators (171), we can write

$$\begin{aligned}
& \langle \Psi_{EM}(1)| \left(|\boldsymbol{k}\rangle\langle \boldsymbol{k}| \otimes \left(a_0 a_0^\dagger - a_3^\dagger a_3\right)_4 \right) \\
\times & \sum_{n_1,n_2=0}^{\infty} \int d\Gamma(\boldsymbol{k}')\ \Psi_{12}(\boldsymbol{k}', n_1, n_2)|\boldsymbol{k}', n_1, n_2\rangle \sum_{n_0,n_3=0}^{\infty} \Psi_{03}(n_3, n_0)|n_3, n_0\rangle \\
= & \langle \Psi_{EM}(1)| \left(I \otimes \left(a_0 a_0^\dagger - a_3^\dagger a_3\right)_4 \right) \sum_{n_1,n_2=0}^{\infty} \Psi_{12}(\boldsymbol{k}, n_1, n_2)|\boldsymbol{k}, n_1, n_2\rangle \sum_{n_0,n_3=0}^{\infty} \Psi_{03}(n_3, n_0)|n_3, n_0\rangle \\
= & \langle \Psi_{12}(1)| \otimes \langle \Psi_{03}(1)| \left(I \otimes 1_2 \otimes \left(a_0 a_0^\dagger - a_3^\dagger a_3\right)_2 \right) \sum_{n_1,n_2=0}^{\infty} \Psi_{12}(\boldsymbol{k}, n_1, n_2)|\boldsymbol{k}, n_1, n_2\rangle \otimes |\Psi_{03}(1)\rangle \\
= & \langle \Psi_{12}(1)|I \otimes 1_2 \sum_{n_1,n_2=0}^{\infty} \Psi_{12}(\boldsymbol{k}, n_1, n_2)|\boldsymbol{k}, n_1, n_2\rangle \times \langle \Psi_{03}(1)| \left(a_0 a_0^\dagger - a_3^\dagger a_3\right)_2 |\Psi_{03}(1)\rangle \\
= & \sum_{n_1,n_2=0}^{\infty} |\Psi_{12}(\boldsymbol{k}, n_1, n_2)|^2 \times \langle \Psi_{03}(1)| \left(a_0 a_0^\dagger - a_3^\dagger a_3\right)_2 |\Psi_{03}(1)\rangle. \qquad (227)
\end{aligned}$$

Assuming that the sum $\sum_{n_1,n_2}^{\infty} |\Psi_{12}(\boldsymbol{k}, n_1, n_2)|^2$ is convergent we can take into account just 0 and 3 polarization degrees of freedom, i.e.

$$\begin{aligned}
& \langle \Psi_{03}(1)|(a_0 a_0^\dagger - a_3^\dagger a_3)_2|\Psi_{03}(1)\rangle \\
= & \sum_{n_0,n_3,n_0',n_3'=0}^{\infty} \langle n_3', n_0'| \frac{e^{-1}}{\sqrt{n_0'! n_3'!}} (a_0 a_0^\dagger - a_3^\dagger a_3) \frac{e^{-1}}{\sqrt{n_0! n_3!}} |n_3, n_0\rangle \\
= & \; e^{-2} \sum_{n_0,n_3=0}^{\infty} \frac{n_0 - n_3}{n_0! n_3!} \; = \; 0.
\end{aligned} \tag{228}$$

For $\Psi_{EM}(1)$ vectors the contribution of time-like photons cancels against the longitudinal ones. Moreover, it can be shown that

$$\langle \Psi_{03}(1)| \,(a_0 - a_3)_2^n \,|\Psi_{03}(1)\rangle \;=\; 0. \tag{229}$$

This is derived explicitly in appendix (D.3).

The extension to an arbitrary N-oscillator representation can be made by

$$|\Psi_{EM}(N)\rangle \;=\; \underbrace{|\Psi_{EM}(1)\rangle \otimes ... \otimes |\Psi_{EM}(1)\rangle}_{N} \;=\; |\Psi_{EM}(1)\rangle^{\otimes N} \;=\; |\Psi_{12}(1)\rangle^{\otimes N} \otimes |\Psi_{03}(1)\rangle^{\otimes N}, \tag{230}$$

so the week Lorenz condition holds also for the N-oscillator representation

$$\langle \Psi_{EM}(N)| \int d\Gamma(\boldsymbol{k}) \,(a_0(\boldsymbol{k}, N) - a_3(\boldsymbol{k}, N))\, |\Psi_{EM}(N)\rangle \;=\; 0. \tag{231}$$

This can be shown by extending formula (218) to N-oscillator representation, having in mind the definition of the ladder operators (78), so that

$$\begin{aligned}
& \langle \Psi_{EM}(N)| \left(\frac{1}{\sqrt{N}} \sum_{n=1}^{N} a_0(\boldsymbol{k}, 1)^{(n)} - \frac{1}{\sqrt{N}} \sum_{m=1}^{N} a_3(\boldsymbol{k}, 1)^{(m)} \right) |\Psi_{EM}(N)\rangle \\
= & \; \frac{1}{\sqrt{N}} \langle \Psi_{EM}(N)| \sum_{n=1}^{N} (a_0(\boldsymbol{k}, 1) - a_3(\boldsymbol{k}, 1))^{(n)} \,|\Psi_{EM}(N)\rangle \\
= & \; \sqrt{N} \langle \Psi_{EM}(1)| \,(a_0(\boldsymbol{k}, 1) - a_3(\boldsymbol{k}, 1))\, |\Psi_{EM}(1)\rangle \times \langle \Psi_{EM}(1)|\Psi_{EM}(1)\rangle^{(N-1)} \;=\; 0.
\end{aligned} \tag{232}$$

Then the number of longitudinal photons equals the number of time-like ones also in the N-oscillator representation

$$\langle \Psi_{EM}(N)| \,(n_0(\boldsymbol{k}, N) - n_3(\boldsymbol{k}, N))\, |\Psi_{EM}(N)\rangle \;=\; 0, \tag{233}$$

since, from the definition of the number operator in N-oscillator representation (89) and the result for $N = 1$ oscillator representation (226), it follows that

$$\begin{aligned}
& \langle \Psi_{EM}(N)| \left(\sum_{n=1}^{N} n_0(\boldsymbol{k}, 1)^{(n)} - \sum_{m=1}^{N} n_3(\boldsymbol{k}, 1)^{(m)} \right) |\Psi_{EM}(N)\rangle \\
= & \; \langle \Psi_{EM}(N)| \sum_{n}^{N} (n_0(\boldsymbol{k}, 1) - n_3(\boldsymbol{k}, 1))^{(n)} \,|\Psi_{EM}(N)\rangle \\
= & \; N \langle \Psi_{EM}(1)| \,(n_0(\boldsymbol{k}, 1) - n_3(\boldsymbol{k}, 1))\, |\Psi_{EM}(1)\rangle \otimes \langle \Psi_{EM}(1)|\Psi_{EM}(1)\rangle^{(N-1)} \;=\; 0.
\end{aligned} \tag{234}$$

4.5 More properties of the potential operator

The covariant structure of ladder operators has its consequence in a covariant commutator of the potential operator taken in arbitrary space-time points:

$$\begin{aligned}
[A_a(x,1), A_b(y,1)] &= \Big[i\int d\Gamma(\boldsymbol{k}) g_a{}^{\boldsymbol{a}}(\boldsymbol{k}) a_{\boldsymbol{a}}(\boldsymbol{k},1) e^{-ik\cdot x} + \text{H.c.}\ , i\int d\Gamma(\boldsymbol{k}') g_b{}^{\boldsymbol{b}}(\boldsymbol{k}') a_{\boldsymbol{b}}(\boldsymbol{k}',1) e^{-ik'\cdot y} + \text{H.c.} \Big] \\
&= -\int d\Gamma(\boldsymbol{k}) \int d\Gamma(\boldsymbol{k}') \Big[g_a{}^{\boldsymbol{a}}(\boldsymbol{k}) a_{\boldsymbol{a}}(\boldsymbol{k},1) e^{-ik\cdot x} - \text{H.c.},\ g_b{}^{\boldsymbol{b}}(\boldsymbol{k}') a_{\boldsymbol{b}}(\boldsymbol{k}',1) e^{-ik'\cdot y} - \text{H.c.} \Big] \\
&= -\int d\Gamma(\boldsymbol{k}) g_a{}^{\boldsymbol{a}}(\boldsymbol{k}) g_b{}^{\boldsymbol{b}}(\boldsymbol{k}) g_{\boldsymbol{ab}} \big(e^{-ik\cdot x} e^{ik\cdot y} - \text{H.c.}\ \big) I(\boldsymbol{k},1) \\
&= -\int d\Gamma(\boldsymbol{k}) g_{ab} (e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)}) I(\boldsymbol{k},1) \\
&= i g_{ab} D(x-y,1). \qquad (235)
\end{aligned}$$

Here g_{ab} is the metric tensor (40), and

$$D(x,1) \ = \ i\int d\Gamma(\boldsymbol{k}) (e^{-ik\cdot x} - e^{ik\cdot x}) I(\boldsymbol{k},1) \qquad (236)$$

is not the usual Jordan Pauli function because at the right-hand side of (236) we have $I(\boldsymbol{k},1)$, the central element of CCR algebra. $D(x,1)$ is Lorentz invariant thanks to Lorentz invariant measure. Furthermore, it has the following properties

$$\begin{aligned}
D(x,1) &= -D(-x,1), \\
D(x,1) &= D^{(+)}(x,1) + D^{(-)}(x,1), \\
D^{(\pm)}(x,1) &= \pm i\int d\Gamma(\boldsymbol{k}) e^{\mp ik\cdot x} I(\boldsymbol{k},1). \qquad (237)
\end{aligned}$$

An extension of the potential operator to N-oscillator representation is equivalent to the N extension of the creation and annihilation operators in (204).

$$\begin{aligned}
A_a(x,N) &= i\int d\Gamma(\boldsymbol{k}) g_a{}^{\boldsymbol{a}}(\boldsymbol{k}) a_{\boldsymbol{a}}(\boldsymbol{k},N) e^{-ik\cdot x} + \text{H.c.} \\
&= i\int d\Gamma(\boldsymbol{k}) \left(-x_a(\boldsymbol{k}) a_1(\boldsymbol{k},N) - y_a(\boldsymbol{k}) a_2(\boldsymbol{k},N) - z_a(\boldsymbol{k}) a_3(\boldsymbol{k},N) + t_a(\boldsymbol{k}) a_0(\boldsymbol{k},N) \right) e^{-ik\cdot x} + \text{H.c.} \\
& \qquad (238)
\end{aligned}$$

For N-oscillator representation the commutator

$$[A_a(x,N), A_b(y,N)] \ = \ i g_{ab} D(x-y,N) \qquad (239)$$

involves again an operator analogue of the Jordan-Pauli function, but this time with the resolution of unity for $\mathcal{H}(N)$ space, i.e.

$$D(x,N) \ = \ i\int d\Gamma(\boldsymbol{k}) \big(e^{-ik\cdot x} - e^{ik\cdot x} \big) I(\boldsymbol{k},N). \qquad (240)$$

To understand why the formalism here constructed is less singular than the one based on irreducible representations, it is instructive to take a closer look at (240). In the first place, formula (240) is typical of all the representations of CCR, reducible or irreducible, and differs only in the central element $I(\boldsymbol{k},N)$. The standard Pauli-Jordan function corresponds to representations where $I(\boldsymbol{k},N)$ equals the identity. Furthermore, it can be split into two parts

$$D(x,N) \ = \ D^{(+)}(x,N) + D^{(-)}(x,N), \qquad (241)$$

$$D^{(\pm)}(x,N) \ = \ \pm i\int d\Gamma(\boldsymbol{k}) I(\boldsymbol{k},N) e^{\mp ik\cdot x} = \frac{1}{N}\sum_{n=1}^{N} D^{(\pm)}(x,1)^{(n)}. \qquad (242)$$

The operator whose N-oscillator extensions occur in (242) reads explicitly

$$D^{(\pm)}(x,1) = \pm i\int d\Gamma(\boldsymbol{k})|\boldsymbol{k}\rangle\langle\boldsymbol{k}|e^{\mp ik\cdot x}\otimes 1_4 = \pm ie^{\mp i\hat{k}\cdot x}\otimes 1_4, \tag{243}$$

$$\hat{k}_{\boldsymbol{a}} = \int d\Gamma(\boldsymbol{k})k_{\boldsymbol{a}}|\boldsymbol{k}\rangle\langle\boldsymbol{k}|. \tag{244}$$

As we can see, the operators $D^{(\pm)}(x,1)$ are unitary representations of four-translations, and their generators are given by $\hat{k}_{\boldsymbol{a}}$. In particular,

$$D^{(\pm)}(0,1) = \pm iI(1), \quad D^{(\pm)}(0,N) = \pm iI(N). \tag{245}$$

As a consequence quantization in terms of this reducible representation replaces distributions $\int d\Gamma(\boldsymbol{k})e^{ik\cdot x}$ with "well behaved" unitary operators $\int d\Gamma(\boldsymbol{k})|\boldsymbol{k}\rangle\langle\boldsymbol{k}|e^{ik\cdot x}$.

For another example, showing advantage of reducible representations, let us consider a single-oscillator four-vector potential operator acting on vacuum state (196)

$$\begin{aligned} A_a(x,1)|O(1)\rangle &= i\left(\int d\Gamma(\boldsymbol{k'})g_a{}^{\boldsymbol{a}}(\boldsymbol{k'})a_{\boldsymbol{a}}(\boldsymbol{k'},1)e^{-ik'\cdot x} - \text{H.c.}\right)|O(1)\rangle & (246)\\ &= \left(i\int d\Gamma(\boldsymbol{k})g_a{}^0(\boldsymbol{k})a_0(\boldsymbol{k})e^{-ik\cdot x} - i\int d\Gamma(\boldsymbol{k})g_a{}^{\boldsymbol{i}}(\boldsymbol{k})a_{\boldsymbol{i}}(\boldsymbol{k})^\dagger e^{ik\cdot x}\right)|O(1)\rangle & (247)\\ &= |A_a(x,1)\rangle. & (248)\end{aligned}$$

A closer look at the scalar product

$$\langle A_a(y,1)|A_b(x,1)\rangle = \int d\Gamma(\boldsymbol{k})\left(g_a{}^0(\boldsymbol{k})g_{b0}(\boldsymbol{k})e^{-ik\cdot(x-y)} + g_a{}^{\boldsymbol{i}}(\boldsymbol{k})g_{b\boldsymbol{i}}(\boldsymbol{k})e^{ik\cdot(x-y)}\right)Z(\boldsymbol{k}) \tag{249}$$

shows that in this type of quantization there is no ultraviolet catastrophe for $x=y$, since $\int d\Gamma(\boldsymbol{k})Z(\boldsymbol{k}) = 1$.

4.6 Electromagnetic field operator

The electromagnetic field operator for $N=1$ oscillator representation is by definition a four-dimensional electromagnetic curl of $A_a(x,1)$

$$F_{ab}(x,1) = \partial_a A_b(x,1) - \partial_b A_a(x,1). \tag{250}$$

This can be written explicitly as

$$\begin{aligned} F_{ab}(x,1) &= \partial_a\left(i\int d\Gamma(\boldsymbol{k})g_b{}^{\boldsymbol{b}}(\boldsymbol{k})a_{\boldsymbol{b}}(\boldsymbol{k},1)e^{-ik\cdot x} + \text{H.c.}\right) - \partial_b\left(i\int d\Gamma(\boldsymbol{k})g_a{}^{\boldsymbol{a}}(\boldsymbol{k})a_{\boldsymbol{a}}(\boldsymbol{k},1)e^{-ik\cdot x} + \text{H.c.}\right)\\ &= i\left(\partial_a\int d\Gamma(\boldsymbol{k})g_b{}^{\boldsymbol{b}}(\boldsymbol{k})a_{\boldsymbol{b}}(\boldsymbol{k},1)e^{-ik\cdot x} - \partial_b\int d\Gamma(\boldsymbol{k})g_a{}^{\boldsymbol{a}}(\boldsymbol{k})a_{\boldsymbol{a}}(\boldsymbol{k},1)e^{-ik\cdot x}\right) + \text{H.c.}\\ &= \int d\Gamma(\boldsymbol{k})\left(k_a(\boldsymbol{k})g_b{}^{\boldsymbol{b}}(\boldsymbol{k})a_{\boldsymbol{b}}(\boldsymbol{k},1) - k_b(\boldsymbol{k})g_a{}^{\boldsymbol{a}}(\boldsymbol{k})a_{\boldsymbol{a}}(\boldsymbol{k},1)\right)e^{-ik\cdot x} + \text{H.c.}\\ &= \int d\Gamma(\boldsymbol{k})\left(k_a(\boldsymbol{k})g_b{}^{\boldsymbol{a}}(\boldsymbol{k}) - k_b(\boldsymbol{k})g_a{}^{\boldsymbol{a}}(\boldsymbol{k})\right)a_{\boldsymbol{a}}(\boldsymbol{k},1)e^{-ik\cdot x} + \text{H.c.}\\ &= \int d\Gamma(\boldsymbol{k})\left(k_a(\boldsymbol{k})t_b(\boldsymbol{k}) - k_b(\boldsymbol{k})t_a(\boldsymbol{k})\right)a_0(\boldsymbol{k},1)e^{-ik\cdot x} + \text{H.c.}\\ &+ \int d\Gamma(\boldsymbol{k})\left(-k_a(\boldsymbol{k})x_b(\boldsymbol{k}) + k_b(\boldsymbol{k})x_a(\boldsymbol{k})\right)a_1(\boldsymbol{k},1)e^{-ik\cdot x} + \text{H.c.}\\ &+ \int d\Gamma(\boldsymbol{k})\left(-k_a(\boldsymbol{k})y_b(\boldsymbol{k}) + k_b(\boldsymbol{k})y_a(\boldsymbol{k})\right)a_2(\boldsymbol{k},1)e^{-ik\cdot x} + \text{H.c.}\\ &+ \int d\Gamma(\boldsymbol{k})\left(-k_a(\boldsymbol{k})z_b(\boldsymbol{k}) + k_b(\boldsymbol{k})z_a(\boldsymbol{k})\right)a_3(\boldsymbol{k},1)e^{-ik\cdot x} + \text{H.c.} \qquad (251)\end{aligned}$$

Having in mind formulas derived in appendices (A.19)–(A.22) one can also write the electromagnetic field operator in terms of spin-frames

$$\begin{aligned}
F_{ab}(x,1) &= \frac{1}{\sqrt{2}}\int d\Gamma(\boldsymbol{k})(\varepsilon_{AB}\pi_{A'}(\boldsymbol{k})\pi_{B'}(\boldsymbol{k})+\varepsilon_{A'B'}\pi_A(\boldsymbol{k})\pi_B(\boldsymbol{k}))a_1(\boldsymbol{k},1)e^{-ik\cdot x}+\text{H.c.}\\
&+ \frac{i}{\sqrt{2}}\int d\Gamma(\boldsymbol{k})(\varepsilon_{AB}\pi_{A'}(\boldsymbol{k})\pi_{B'}(\boldsymbol{k})-\varepsilon_{A'B'}\pi_A(\boldsymbol{k})\pi_B(\boldsymbol{k}))a_2(\boldsymbol{k},1)e^{-ik\cdot x}+\text{H.c.}\\
&+ \frac{1}{\sqrt{2}}\int d\Gamma(\boldsymbol{k})(\omega_a(\boldsymbol{k})k_b(\boldsymbol{k})-k_a(\boldsymbol{k})\omega_b(\boldsymbol{k}))a_3(\boldsymbol{k},1)e^{-ik\cdot x}+\text{H.c.}\\
&- \frac{1}{\sqrt{2}}\int d\Gamma(\boldsymbol{k})(\omega_a(\boldsymbol{k})k_b(\boldsymbol{k})-k_a(\boldsymbol{k})\omega_b(\boldsymbol{k}))a_0(\boldsymbol{k},1)e^{-ik\cdot x}+\text{H.c.}\\
&= \int d\Gamma(\boldsymbol{k})\pi_A(\boldsymbol{k})\pi_B(\boldsymbol{k})\varepsilon_{A'B'}\left(a_-(\boldsymbol{k},1)e^{-ik\cdot x}+a_+(\boldsymbol{k},1)^\dagger e^{ik\cdot x}\right)+\text{H.c.}\\
&+ \frac{1}{\sqrt{2}}\int d\Gamma(\boldsymbol{k})^*M_{ab}(\boldsymbol{k})\left(a_3(\boldsymbol{k},1)-a_0(\boldsymbol{k},1)\right)e^{-ik\cdot x}+\text{H.c.}
\end{aligned}\tag{252}$$

Here

$$^*M_{ab}(\boldsymbol{k}) = \omega_a(\boldsymbol{k})k_b(\boldsymbol{k})-\omega_b(\boldsymbol{k})k_a(\boldsymbol{k})\tag{253}$$

is a tensor dual to massless angular momentum tensor:

$$M_{ab}(\boldsymbol{k}) = i\pi_{(A}(\boldsymbol{k})\omega_{B)}(\boldsymbol{k})\varepsilon_{A'B'}-i\pi_{(A'}(\boldsymbol{k})\omega_{B')}(\boldsymbol{k})\varepsilon_{AB}.\tag{254}$$

As we can see the field tensor (252) splits in two parts: the first part involves transverse photons, while the second one consists of a "bad ghost" operator corresponding to particles unmeasured in experiments. It can be shown that for $\Psi_{EM}(1)$ vectors the electromagnetic field operator corresponds to standard electromagnetic theory

$$\begin{aligned}
&\langle\Psi_{EM}(1)|F_{ab}(x,1)|\Psi_{EM}(1)\rangle\\
&= \langle\Psi_{EM}(1)|\left(\int d\Gamma(\boldsymbol{k})\pi_A(\boldsymbol{k})\pi_B(\boldsymbol{k})\varepsilon_{A'B'}\left(a_-(\boldsymbol{k},1)e^{-ik\cdot x}+a_+(\boldsymbol{k},1)^\dagger e^{ik\cdot x}\right)+\text{H.c.}\right)|\Psi_{EM}(1)\rangle
\end{aligned}\tag{255}$$

Let us also check the Maxwell equations for the electromagnetic field operator

$$\begin{aligned}
&\partial_c F_{ab}(x,1)+\partial_a F_{bc}(x,1)+\partial_b F_{ca}(x,1)\\
&= -i\int d\Gamma(\boldsymbol{k})\left(k_c(\boldsymbol{k})k_a(\boldsymbol{k})g_b{}^{\boldsymbol{a}}(\boldsymbol{k})-k_c(\boldsymbol{k})k_b(\boldsymbol{k})g_a{}^{\boldsymbol{a}}(\boldsymbol{k})\right)a_{\boldsymbol{a}}(\boldsymbol{k},1)e^{-ik\cdot x}+\text{H.c.}\\
&- i\int d\Gamma(\boldsymbol{k})\left(k_a(\boldsymbol{k})k_b(\boldsymbol{k})g_c{}^{\boldsymbol{a}}(\boldsymbol{k})-k_a(\boldsymbol{k})k_c(\boldsymbol{k})g_b{}^{\boldsymbol{a}}(\boldsymbol{k})\right)a_{\boldsymbol{a}}(\boldsymbol{k},1)e^{-ik\cdot x}+\text{H.c.}\\
&- i\int d\Gamma(\boldsymbol{k})\left(k_b(\boldsymbol{k})k_c(\boldsymbol{k})g_a{}^{\boldsymbol{a}}(\boldsymbol{k})-k_b(\boldsymbol{k})k_a(\boldsymbol{k})g_c{}^{\boldsymbol{a}}(\boldsymbol{k})\right)a_{\boldsymbol{a}}(\boldsymbol{k},1)e^{-ik\cdot x}+\text{H.c.}\\
&= 0,
\end{aligned}\tag{256}$$

$$\begin{aligned}
\partial_a F^{ab}(x,1) &= -ik_a\int d\Gamma(\boldsymbol{k})\left(k^a(\boldsymbol{k})t^b(\boldsymbol{k})-k^b(\boldsymbol{k})t^a(\boldsymbol{k})\right)a_0(\boldsymbol{k},1)e^{-ik\cdot x}+\text{H.c.}\\
&- ik_a\int d\Gamma(\boldsymbol{k})\left(-k^a(\boldsymbol{k})x^b(\boldsymbol{k})+k^b(\boldsymbol{k})x^a(\boldsymbol{k})\right)a_1(\boldsymbol{k},1)e^{-ik\cdot x}+\text{H.c.}\\
&- ik_a\int d\Gamma(\boldsymbol{k})\left(-k^a(\boldsymbol{k})y^b(\boldsymbol{k})+k^b(\boldsymbol{k})y^a(\boldsymbol{k})\right)a_2(\boldsymbol{k},1)e^{-ik\cdot x}+\text{H.c.}\\
&- ik_a\int d\Gamma(\boldsymbol{k})\left(-k^a(\boldsymbol{k})z^b(\boldsymbol{k})+k^b(\boldsymbol{k})z^a(\boldsymbol{k})\right)a_3(\boldsymbol{k},1)e^{-ik\cdot x}+\text{H.c.}\\
&= \frac{i}{\sqrt{2}}\int d\Gamma(\boldsymbol{k})k^b(\boldsymbol{k})\left(a_0(\boldsymbol{k},1)-a_3(\boldsymbol{k},1)\right)e^{-ik\cdot x}+\text{H.c.}
\end{aligned}\tag{257}$$

The second equation (257) does not correspond to standard Maxwell electromagnetism theory, but it can be shown that in $\Psi_{EM}(1)$ averages

$$\langle\Psi_{EM}(1)|\partial_a F^{ab}(x,1)|\Psi_{EM}(1)\rangle = 0.\tag{258}$$

As in the potential operator, extension of the electromagnetic field to N-oscillator representation is equivalent to the extension to arbitrary N of creation and annihilation operators in (251).

4.7 Coherent states

Coherent states were introduced in 1963 by Glauber, furthermore, he received in 2005 the Nobel prize for his work in this direction. In general, coherent states have the property of minimizing an uncertainty principle, which means that they are closest to classical states. The mathematical structure of coherent states for two-photon polarization degrees of freedom is well known form the literature. Here the construction will be extended to the reducible representation and to the two additional, time-like and longitudinal, polarization degrees of freedom. It turns out that such an abstract structure has its interpretation also in terms of the Ψ_{EM} state. This will be investigated further in the next section.

Now let us start from the $N = 1$ oscillator representation. In this case the displacement operator for four-dimensional oscillator algebra will be defined as

$$\mathcal{D}(\alpha, 1) = \exp\left(\int d\Gamma(\boldsymbol{k})\Big(\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}a_{\boldsymbol{a}}(\boldsymbol{k}, 1) - \text{H.c.}\Big)\right). \tag{259}$$

Here $\alpha^{\boldsymbol{a}}(\boldsymbol{k})$ is a function corresponding to the "amount of displacement" and can depend on $\boldsymbol{k}$. Acting with the operator (259) on a vacuum state we get a coherent state

$$\mathcal{D}(\alpha, 1)|O(1)\rangle = |\alpha(1)\rangle. \tag{260}$$

Furthermore, coherent states for $N = 1$ oscillator representation can be explicitly expressed as

$$\begin{aligned} |\alpha(1)\rangle &= \int d\Gamma(\boldsymbol{k})O(\boldsymbol{k})\exp\left(-\frac{1}{2}\left(|\alpha_1(\boldsymbol{k})|^2 + |\alpha_2(\boldsymbol{k})|^2 + |\alpha_3(\boldsymbol{k})|^2 + |\alpha_0(\boldsymbol{k})|^2\right)\right) \\ &\sum_{n_1,n_2,n_3,n_0=0}^{\infty} \frac{(\alpha_1(\boldsymbol{k}))^{n_1}(\alpha_2(\boldsymbol{k}))^{n_2}(\alpha_3(\boldsymbol{k}))^{n_3}(\overline{\alpha_0(\boldsymbol{k})})^{n_0}}{\sqrt{n_1!n_2!n_3!n_0!}}|\boldsymbol{k}, n_1, n_2, n_3, n_0\rangle. \end{aligned} \tag{261}$$

This is derived step by step in appendix (G.3). Acting on vacuum with N-oscillator representation of the covariant displacement operator defined as

$$\begin{aligned} \mathcal{D}(\alpha, N) &= \exp\left(\int d\Gamma(\boldsymbol{k})\Big(\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}a_{\boldsymbol{a}}(\boldsymbol{k}, N) - \text{H.c.}\Big)\right) \\ &= \exp\left(\int d\Gamma(\boldsymbol{k})\Big(\overline{\alpha^1(\boldsymbol{k})}a_1(\boldsymbol{k}, N) + \overline{\alpha^2(\boldsymbol{k})}a_2(\boldsymbol{k}, N) + \overline{\alpha^3(\boldsymbol{k})}a_3(\boldsymbol{k}, N) + \overline{\alpha^0(\boldsymbol{k})}a_0(\boldsymbol{k}, N) - \text{H.c.}\Big)\right) \end{aligned} \tag{262}$$

we obtain a coherent state for the N-oscillator representation

$$|\alpha(N)\rangle = \mathcal{D}(\alpha, N)|O(N)\rangle. \tag{263}$$

It is also shown in appendix (G.9) that

$$\begin{aligned} \mathcal{D}(\alpha, N) &= \exp\left(\int d\Gamma(\boldsymbol{k})\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}a_{\boldsymbol{a}}(\boldsymbol{k}, N)\right)\exp\left(-\int d\Gamma(\boldsymbol{k})\alpha^{\boldsymbol{a}}(\boldsymbol{k})a_{\boldsymbol{a}}(\boldsymbol{k}, N)^\dagger\right) \\ &\times \exp\left(\frac{1}{2}\int d\Gamma(\boldsymbol{k})\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}\alpha_{\boldsymbol{a}}(\boldsymbol{k})I(\boldsymbol{k}, N)\right). \end{aligned} \tag{264}$$

The covariant displacement operator in N-oscillator representation can be also written in terms of the $N = 1$ representation as follows

$$\mathcal{D}(\alpha, N) = \exp\left(\int d\Gamma(\boldsymbol{k})\Big(\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}\frac{1}{\sqrt{N}}a_{\boldsymbol{a}}(\boldsymbol{k}, 1) - \text{H.c.}\Big)\right)^{\otimes N} = \mathcal{D}\left(\frac{\alpha}{\sqrt{N}}, 1\right)^{\otimes N}. \tag{265}$$

This formula is derived step by step in appendix (G.4). Also the product of two displacement operators is another displacement operator, apart from an operator valued phase factor, which does not contribute to expectation values, i.e.

$$\mathcal{D}(\alpha, N)\mathcal{D}(\beta, N) = \mathcal{D}(\alpha + \beta, N)\exp\left(\frac{1}{2}\int d\Gamma(\boldsymbol{k})\Big(\overline{\alpha_{\boldsymbol{a}}(\boldsymbol{k})}\beta^{\boldsymbol{a}}(\boldsymbol{k}) - \overline{\beta_{\boldsymbol{a}}(\boldsymbol{k})}\alpha^{\boldsymbol{a}}(\boldsymbol{k})\Big)I(\boldsymbol{k}, N)\right). \tag{266}$$

From this it can be shown that the displacement operator is unitary

$$\mathcal{D}(\alpha, N)^\dagger \;=\; \mathcal{D}(-\alpha, N) \;=\; \mathcal{D}(\alpha, N)^{-1}. \tag{267}$$

Then, the inner product of two coherent states is normalized, i.e.

$$\langle \alpha(N)|\alpha(N)\rangle \;=\; \langle O(N)|\mathcal{D}(\alpha, N)^\dagger \mathcal{D}(\alpha, N)|O(N)\rangle \;=\; \langle O(N)|O(N)\rangle \;=\; 1. \tag{268}$$

The displacement operator in these representations shifts the creation and annihilation operators, but leaves the central elements unchanged. This is shown step by step in appendix (G.12)-(G.13):

$$\begin{aligned}
\mathcal{D}(\alpha, N)^\dagger a_{\boldsymbol{a}}(\boldsymbol{k}, N)\mathcal{D}(\alpha, N) &= a_{\boldsymbol{a}}(\boldsymbol{k}, N) + \alpha_{\boldsymbol{a}}(\boldsymbol{k}) I(\boldsymbol{k}, N), &(269)\\
\mathcal{D}(\alpha, N)^\dagger a_{\boldsymbol{a}}(\boldsymbol{k}, N)^\dagger \mathcal{D}(\alpha, N) &= a_{\boldsymbol{a}}(\boldsymbol{k}, N)^\dagger + \overline{\alpha_{\boldsymbol{a}}(\boldsymbol{k})} I(\boldsymbol{k}, N), &(270)\\
\mathcal{D}(\alpha, N)^\dagger I(\boldsymbol{k}, N)\mathcal{D}(\alpha, N) &= I(\boldsymbol{k}, N). &(271)
\end{aligned}$$

A generalized eigenvalue problem for coherent states in this representation can be formed. Here in reducible representation coherent states are not just eigenstates of annihilation operators. To see this let us take a closer look at the lowering energy operators acting on coherent states

$$\begin{aligned}
a_{\boldsymbol{a}}(\boldsymbol{k}, N)|\alpha(N)\rangle &= a_{\boldsymbol{a}}(\boldsymbol{k}, N)\mathcal{D}(\alpha, N)|O(N)\rangle\\
&= \mathcal{D}(\alpha, N)\mathcal{D}(\alpha, N)^\dagger a_{\boldsymbol{a}}(\boldsymbol{k}, N)\mathcal{D}(\alpha, N)|O(N)\rangle\\
&= \mathcal{D}(\alpha, N)\left(a_{\boldsymbol{a}}(\boldsymbol{k}, N) + \alpha_{\boldsymbol{a}}(\boldsymbol{k}) I(\boldsymbol{k}, N)\right)|O(N)\rangle. &(272)
\end{aligned}$$

Only operators from the $\boldsymbol{j} = 1, 2, 3$ polarization degree of freedom are annihilation operators, so

$$a_{\boldsymbol{j}}(\boldsymbol{k}, N)|\alpha(N)\rangle \;=\; \alpha_{\boldsymbol{j}}(\boldsymbol{k}) I(\boldsymbol{k}, N)|\alpha(N)\rangle, \quad \boldsymbol{j} = 1, 2, 3. \tag{273}$$

As one can see $\alpha_{\boldsymbol{j}}(\boldsymbol{k})$ is not an eigenvalue of the annihilation operator alone due to the $I(\boldsymbol{k}, N)$ operator. We can follow the same procedure for raising energy operators

$$\begin{aligned}
a_{\boldsymbol{a}}(\boldsymbol{k}, N)^\dagger|\alpha(N)\rangle &= a_{\boldsymbol{a}}(\boldsymbol{k}, N)^\dagger \mathcal{D}(\alpha, N)|O(N)\rangle\\
&= \mathcal{D}(\alpha, N)\mathcal{D}(\alpha, N)^\dagger a_{\boldsymbol{a}}(\boldsymbol{k}, N)^\dagger \mathcal{D}(\alpha, N)|O(N)\rangle\\
&= \mathcal{D}(\alpha, N)\left(a_{\boldsymbol{a}}(\boldsymbol{k}, N)^\dagger + \overline{\alpha_{\boldsymbol{a}}(\boldsymbol{k})} I(\boldsymbol{k}, N)\right)|O(N)\rangle. &(274)
\end{aligned}$$

Having in mind that $a_0(\boldsymbol{k}, N)^\dagger$ annihilates vacuum we can write

$$a_0(\boldsymbol{k}, N)^\dagger|\alpha(N)\rangle \;=\; \overline{\alpha_0(\boldsymbol{k})} I(\boldsymbol{k}, N)|\alpha(N)\rangle. \tag{275}$$

Also one can form a generalized eigenvalue problem for the annihilation operators within the whole frequency spectrum. This is a result of the assumption that $\alpha_{\boldsymbol{a}}(\boldsymbol{k})$ can in general be $\boldsymbol{k}$ dependent, so that

$$\begin{aligned}
a_{\boldsymbol{j}}(N)|\alpha(N)\rangle &= \int d\Gamma(\boldsymbol{k})\ \alpha_{\boldsymbol{j}}(\boldsymbol{k}) I(\boldsymbol{k}, N)|\alpha(N)\rangle, \quad \boldsymbol{j} = 1, 2, 3, &(276)\\
a_0(N)^\dagger|\alpha(N)\rangle &= \int d\Gamma(\boldsymbol{k})\ \overline{\alpha_0(\boldsymbol{k})} I(\boldsymbol{k}, N)|\alpha(N)\rangle. &(277)
\end{aligned}$$

When displacement operators act on fields, the field operators get shifted by the central elements, i.e.

$$\begin{aligned}
\mathcal{D}(\alpha, N)^\dagger A_a(x, N)\mathcal{D}(\alpha, N) &= A_a(x, N) + i\int d\Gamma(\boldsymbol{k}) I(\boldsymbol{k}, N) g_a{}^{\boldsymbol{a}}(\boldsymbol{k})\alpha_{\boldsymbol{a}}(\boldsymbol{k}) e^{-ik\cdot x} + \text{H.c.}\\
&= A_a(x, N) + \hat{A}_a(\alpha, x). &(278)
\end{aligned}$$

It should be stressed that the shift $\hat{A}_a(\alpha, x)$ is not a classical field but an element of the center of the CCR algebra.

4.8 Displacement-like operator for the Ψ_{EM} states

As seen before in section 4.4, Ψ_{EM} states have coherent-like structure. Therefore, it would be good to investigate this structure in more detail. For this purpose let us define a displacement-like operator for time-like and longitudinal degrees of photon polarization. We will start from the $N = 1$ oscillator representation denoting

$$\mathcal{D}_{03}(1) \quad = \quad \exp\left(\int d\Gamma(\boldsymbol{k})\Big(\overline{\alpha^0(\boldsymbol{k})}a_0(\boldsymbol{k},1) + \overline{\alpha^3(\boldsymbol{k})}a_3(\boldsymbol{k},1) - \text{H.c.}\Big)\right). \tag{279}$$

Acting with operator (279) on a vacuum state we get a vector state of the form

$$\begin{aligned} & \mathcal{D}_{03}(\alpha,1)|O(1)\rangle \\ = & \sum_{n_0,n_3=0}^{\infty}\int d\Gamma(\boldsymbol{k})O(\boldsymbol{k})\exp\left(-\frac{1}{2}\left(|\alpha_3(\boldsymbol{k})|^2+|\alpha_0(\boldsymbol{k})|^2\right)\right)\frac{(\alpha_3(\boldsymbol{k}))^{n_3}\,\overline{(\alpha_0(\boldsymbol{k}))^{n_0}}}{\sqrt{n_3!n_0!}}|\boldsymbol{k},n_1,n_2,n_3,n_0\rangle \\ = & |\alpha_{03}(1)\rangle. \end{aligned} \tag{280}$$

This state has a more general form than the previous Ψ_{EM} from section 4.4, because of the amplitudes $\alpha_3(\boldsymbol{k}),\ \alpha_0(\boldsymbol{k})$. It can be extended to N-oscillator representation by

$$\mathcal{D}_{03}(\alpha,N) \quad = \quad \exp\left(\int d\Gamma(\boldsymbol{k})\Big(\overline{\alpha^0(\boldsymbol{k})}a_0(\boldsymbol{k},N) + \overline{\alpha^3(\boldsymbol{k})}a_3(\boldsymbol{k},N) - \text{H.c.}\Big)\right), \tag{281}$$

$$\mathcal{D}_{03}(\alpha,N)|O(N)\rangle \quad = \quad |\alpha_{03}(N)\rangle. \tag{282}$$

Now, let us check the weaker Lorenz condition for such states

$$\begin{aligned} & \langle\alpha_{03}(N)|a_0(\boldsymbol{k},N) - a_3(\boldsymbol{k},N)|\alpha_{03}(N)\rangle \\ = & \langle O(N)|\mathcal{D}_{03}(\alpha,N)^{\dagger}\left(a_0(\boldsymbol{k},N)-a_3(\boldsymbol{k},N)\right)\mathcal{D}_{03}(\alpha,N)|O(N)\rangle \\ = & \langle O(N)|a_0(\boldsymbol{k},N)+\alpha_0(\boldsymbol{k})I(\boldsymbol{k},N)-a_3(\boldsymbol{k},N)-\alpha_3(\boldsymbol{k})I(\boldsymbol{k},N)|O(N)\rangle \\ = & \langle O(N)|\left(\alpha_0(\boldsymbol{k})-\alpha_3(\boldsymbol{k})\right)I(\boldsymbol{k},N)|O(N)\rangle. \end{aligned} \tag{283}$$

So, if we want the ghost operator to vanish, we have to fulfill the condition

$$\alpha_0(\boldsymbol{k}) \ = \ \alpha_3(\boldsymbol{k}) \ = \ \alpha(\boldsymbol{k}), \tag{284}$$

so that

$$\mathcal{D}_{EM}(\alpha,N) \quad = \quad \exp\left(\int d\Gamma(\boldsymbol{k})\Big(\overline{\alpha(\boldsymbol{k})}\left(a_0(\boldsymbol{k},N)+a_3(\boldsymbol{k},N)\right) - \text{H.c.}\Big)\right), \tag{285}$$

$$|\Psi_{EM}(N)\rangle \quad = \quad \mathcal{D}_{EM}(\alpha,N)|O(N)\rangle. \tag{286}$$

Of course, like the displacement operator, $\mathcal{D}_{EM}(\alpha,N)$ is unitary:

$$\mathcal{D}_{EM}(\alpha,N)^{\dagger} \ = \ \mathcal{D}_{EM}(-\alpha,N) \ = \ \mathcal{D}_{EM}(\alpha,N)^{-1}, \tag{287}$$

so the inner product of two $\Psi_{EM}(N)$ states is normalized, i.e.

$$\langle\Psi_{EM}(N)|\Psi_{EM}(N)\rangle \ = \ \langle O(N)|\mathcal{D}_{EM}(\alpha,N)^{\dagger}\mathcal{D}_{EM}(\alpha,N)|O(N)\rangle \ = \ \langle O(N)|O(N)\rangle \ = \ 1. \tag{288}$$

Moreover, following the result for coherent states form the previous section (265), we may write explicitly that

$$|\Psi_{EM}(N)\rangle \quad = \quad \left(\sum_{n_3,n_0=0}^{\infty}\int d\Gamma(\boldsymbol{k})O(\boldsymbol{k})\exp\left(-\frac{1}{N}|\alpha(\boldsymbol{k})|^2\right)\frac{\left(\frac{\alpha(\boldsymbol{k})}{\sqrt{N}}\right)^{n_3}\overline{\left(\frac{\alpha(\boldsymbol{k})}{\sqrt{N}}\right)^{n_0}}}{\sqrt{n_3!n_0!}}|\boldsymbol{k},n_1,n_2,n_3,n_0\rangle\right)^{\otimes N}. \tag{289}$$

4.9 Conclusions and results

In this chapter reducible representations of the four-dimensional polarization oscillators were presented. Using the covariant Hamiltonian (194) for N-oscillator representation, we find out that such a formalism is free from vacuum energy divergences. The convergence of vacuum energy is guaranteed by the proper choice of the vacuum probability density function $Z(\boldsymbol{k})$. Furthermore, the analysis shows that the parameter N may even be a finite number for such representations. The same four-dimensional polarization quantization as in this chapter was already formulated by Czachor and Naudts in [12] and further by Czachor and Wrzask in [13], where in the place of the annihilation operator of Gupta-Bleuler-type potential for the time-like degree of freedom, stands a creation operator and vice-versa. The new analysis from this chapter follows from the definition of a covariant Hamiltonian (108) and is in agreement with such a choice of quantization.

Further sections 4.4 and 4.8 contain new results, deriving Ψ_{EM} vectors reproducing standard electromagnetic fields (i.e. photons with two polarization degrees of freedom) from the four-dimensional covariant formalism (i.e. with two additional, longitudinal and time-like polarizations). It is interesting that such vectors have a coherent-like structure.

5 Lorentz transformation

In this chapter a homogeneous Lorentz transformation for the four-dimensional oscillator in reducible representation is introduced. When taking into account the four photon polarization degrees of freedom, the Lorentz transformation is accompanied by another transformation and this manifests itself also on the spin-frame level. We will start from the irreducible representation. First in section 5.1 two transformations that leave the four-momentum invariant on the spin-frame level are introduced: the Wigner rotations and a transformation that will manifest itself as the gauge transformation. In section 5.2 an SL(2,C) transformation matrix is introduced. In 5.3 an explicit transformation of tetrads is shown. In 5.4 a corresponding transformation of the ladder operators is derived. Further in section 5.5 the generators for the irreducible representation are introduced. Finally in section 5.6 one introduces Lorentz transformations in our reducible representation. In section 5.7 the composition law for Lorentz transformation is proved. Further in section 5.8 transformation properties of the vector potential are shown. In 5.9 those of the electromagnetic field operator, and in 5.10 those of vacuum are discussed. In section 5.11 it is shown that there exists a transformation on the spin-frame level that corresponds to a gauge transformation of the four-vector potential. In section 5.12 it is shown that the "ghost" operator and the covariant number operator are invariant in any gauge and in any reference frame. Finally in section 5.13 four translations in the four-dimensional oscillator representation will be introduced.

5.1 Transformation properties of spin-frames

On the spin-frame level there exist two symmetries that leave the spin-frame condition (13) invariant. First, the spinor field transformation associated with the homogeneous Lorentz transformation

$$\pi_A(\boldsymbol{k}) \quad \mapsto \quad \Lambda\pi_A(\boldsymbol{k}) \;=\; \Lambda_A{}^B\pi_B(\boldsymbol{\Lambda^{-1}k}). \tag{290}$$

Here $\boldsymbol{\Lambda^{-1}k}$ is a space-like part of a four-vector $\Lambda^{-1}{}_a{}^b k_b(\boldsymbol{k})$ and $\Lambda_A{}^B$ is an unprimed SL(2,C) matrix corresponding to $\Lambda_a{}^b \in$ SO(1,3). The null vector $k_a(\boldsymbol{k})$ that plays the role of a flag-pole for the spinor field $\pi_A(\boldsymbol{k})$, i.e. $k_a(\boldsymbol{k}) = \pi_A(\boldsymbol{k})\pi_{A'}(\boldsymbol{k})$, must be invariant, so $\Lambda\pi_A(\boldsymbol{k})\Lambda\pi_{A'}(\boldsymbol{k}) = \pi_A(\boldsymbol{k})\pi_{A'}(\boldsymbol{k})$ must be satisfied and hence

$$\Lambda\pi_A(\boldsymbol{k}) \;=\; e^{-i\Theta(\Lambda,\boldsymbol{k})}\pi_A(\boldsymbol{k}). \tag{291}$$

The angle $\Theta(\Lambda,\boldsymbol{k})$ is called the Wigner phase. Note that in the literature it is the doubled value $2\Theta(\Lambda,\boldsymbol{k})$ which is called the Wigner phase. In analogy

$$\omega_A(\boldsymbol{k}) \quad \mapsto \quad \Lambda\omega_A(\boldsymbol{k}) \;=\; \Lambda_A{}^B\omega_B(\boldsymbol{\Lambda^{-1}k}), \tag{292}$$

and the spin-frame condition has to hold. We assume a special case, i.e.

$$\Lambda\omega_A(\boldsymbol{k}) \;=\; e^{i\Theta(\Lambda,\boldsymbol{k})}\omega_A(\boldsymbol{k}). \tag{293}$$

It is possible to find such a spin-frame, and this was discussed in [13] paper by Czachor and Wrzask, where the oscillators are characterized by an additional center-of mass $\boldsymbol{R}$ coordinate. However such new coordinate does not bring anything essential to the discussion and therefore will not be used here. Furthermore, it is important to stress that the Wigner phase depends only on the direction of the momentum and does not depend on the frequency, so that all the parallel wave vectors correspond to the same rotational angle. This was shown, for example, by Caban and Rembieliński in [70].

Let us define another symmetry

$$\omega_A(\boldsymbol{k}) \quad \mapsto \quad \tilde{\omega}_A(\boldsymbol{k}) \;=\; \omega_A(\boldsymbol{k}) + \phi(\boldsymbol{k})\pi_A(\boldsymbol{k}), \tag{294}$$

$$\pi_A(\boldsymbol{k}) \quad \mapsto \quad \tilde{\pi}_A(\boldsymbol{k}) \;=\; \pi_A(\boldsymbol{k}), \tag{295}$$

which also keeps the spin-frame condition (13). Here $\phi(\boldsymbol{k}) = |\phi(\boldsymbol{k})|e^{i\xi(\boldsymbol{k})}$ is at this point any complex number. It is interesting that the ambiguity of $\phi(\boldsymbol{k})$ at the spinor level manifests itself in electrodynamics. This problem will be discussed further in section 5.11. It turns out that the freedom of choosing $\phi(\boldsymbol{k})$ on the spin-frame level, is equivalent to a gauge freedom for the four-vector potential.

The spin-frame condition

$$\Lambda\tilde{\omega}_A(\boldsymbol{k})\Lambda\pi^A(\boldsymbol{k}) \;=\; 1 \tag{296}$$

holds for the most general transformation written as

$$\Lambda\tilde{\omega}_A(\boldsymbol{k}) \;=\; e^{i\Theta(\Lambda,\boldsymbol{k})}\left(\omega_A(\boldsymbol{k})+\phi(\boldsymbol{k})\pi_A(\boldsymbol{k})\right) \;=\; e^{i\Theta(\Lambda,\boldsymbol{k})}\tilde{\omega}_A(\boldsymbol{k}). \tag{297}$$

Also

$$\Lambda\tilde{\omega}_A(\boldsymbol{k}) \;=\; \Lambda_A{}^B\omega_B(\boldsymbol{\Lambda^{-1}k})+\Lambda_A{}^B\phi(\boldsymbol{\Lambda^{-1}k})\pi_B(\boldsymbol{\Lambda^{-1}k}) \;=\; e^{i\Theta(\Lambda,\boldsymbol{k})}\left(\omega_A(\boldsymbol{k})+\phi(\boldsymbol{\Lambda^{-1}k})e^{-2i\Theta(\Lambda,\boldsymbol{k})}\pi_A(\boldsymbol{k})\right). \tag{298}$$

So, the transformation rule for $\phi(\boldsymbol{k})$ under Lorentz transformation is

$$\Lambda\phi(\boldsymbol{k}) \;=\; \phi(\boldsymbol{\Lambda^{-1}k}) \;=\; e^{2i\Theta(\Lambda,\boldsymbol{k})}\phi(\boldsymbol{k}). \tag{299}$$

Moreover, it will be shown later in section 5.7 that such a transformation rule satisfies the composition law, i.e.

$$\Lambda\Lambda'\phi(\boldsymbol{k}) \;=\; \phi((\boldsymbol{\Lambda\Lambda'})^{-1}\boldsymbol{k}) \;=\; e^{2i\Theta(\Lambda\Lambda',\boldsymbol{k})}\phi(\boldsymbol{k}). \tag{300}$$

It should be stressed that there is a difference in interpretation of $\phi(\boldsymbol{k})$ compared with papers [12] and [13]. Here the Lorentz transformation is not parameterized by $\phi(\boldsymbol{k})$ and this implies a difference in (297) notation, when compared with $\Lambda\omega_A(\boldsymbol{k}) = e^{i\Theta(\Lambda,\boldsymbol{k})}\left(\omega_A(\boldsymbol{k})+\phi(\boldsymbol{k})\pi_A(\boldsymbol{k})\right)$ from [12] and [13] papers. This way the succession of these two transformations is emphasised, because it is important to stress that Lorentz transformations and the transformation parameterized by $\phi(\boldsymbol{k})$ do not commute.

5.2 SL(2,C) transformation matrix

Now, for any homogeneous Lorentz transformation $\Lambda_a{}^b$, parameterized by the Wigner phase $\Theta(\Lambda,\boldsymbol{k})$, and any gauge transformation parameterized by some complex number $\phi(\boldsymbol{k})$, let us define the following matrix associated with the Minkowski tetrad (33)

$$L_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Theta,\phi) \;=\; g^a{}_{\boldsymbol{a}}(\boldsymbol{k})\Lambda_a{}^b\tilde{g}_b{}^{\boldsymbol{b}}(\boldsymbol{\Lambda^{-1}k}) \;=\; g^a{}_{\boldsymbol{a}}(\boldsymbol{k})\Lambda\tilde{g}_a{}^{\boldsymbol{b}}(\boldsymbol{k}). \tag{301}$$

This matrix has the property of linking two Minkowski tetrads in a way that

$$g_c{}^{\boldsymbol{a}}(\boldsymbol{k})L_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Theta,\phi) \;=\; g_c{}^{\boldsymbol{a}}(\boldsymbol{k})g^a{}_{\boldsymbol{a}}(\boldsymbol{k})\Lambda_a{}^b\tilde{g}_b{}^{\boldsymbol{b}}(\boldsymbol{\Lambda^{-1}k}) \;=\; g_c{}^a\Lambda_a{}^b\tilde{g}_b{}^{\boldsymbol{b}}(\boldsymbol{\Lambda^{-1}k}) \;=\; \Lambda_c{}^b\tilde{g}_b{}^{\boldsymbol{b}}(\boldsymbol{\Lambda^{-1}k}). \tag{302}$$

The metric tensor must be invariant under this combined Lorentz and gauge transformation, therefore

$$g_a{}^{\boldsymbol{a}}(\boldsymbol{k})g_b{}^{\boldsymbol{b}}(\boldsymbol{k})g_{\boldsymbol{ab}} \;=\; g_a{}^{\boldsymbol{c}}(\boldsymbol{k})L_{\boldsymbol{c}}{}^{\boldsymbol{a}}(\Theta,\phi)g_b{}^{\boldsymbol{d}}(\boldsymbol{k})L_{\boldsymbol{d}}{}^{\boldsymbol{b}}(\Theta,\phi)g_{\boldsymbol{ab}} \;=\; g_a{}^{\boldsymbol{c}}(\boldsymbol{k})g_b{}^{\boldsymbol{d}}(\boldsymbol{k})L_{\boldsymbol{c}}{}^{\boldsymbol{a}}(\Theta,\phi)L_{\boldsymbol{da}}(\Theta,\phi). \tag{303}$$

Furthermore, this implies that

$$L_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Theta,\phi)L_{\boldsymbol{cb}}(\Theta,\phi) \;=\; L_{\boldsymbol{ab}}(\Theta,\phi)L_{\boldsymbol{c}}{}^{\boldsymbol{b}}(\Theta,\phi) \;=\; g_{\boldsymbol{ac}}, \tag{304}$$
$$L_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Theta,\phi)L^{\boldsymbol{c}}{}_{\boldsymbol{b}}(\Theta,\phi) \;=\; L_{\boldsymbol{ab}}(\Theta,\phi)L^{\boldsymbol{cb}}(\Theta,\phi) \;=\; g_{\boldsymbol{a}}{}^{\boldsymbol{c}}, \tag{305}$$
$$L^{-1}{}_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Theta,\phi) \;=\; L^{\boldsymbol{b}}{}_{\boldsymbol{a}}(\Theta,\phi). \tag{306}$$

From (301) one can get

$$L_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Theta,\phi) \;=\; g^a{}_{\boldsymbol{a}}(\boldsymbol{k})\Lambda_a{}^b\tilde{g}_b{}^{\boldsymbol{b}}(\boldsymbol{\Lambda^{-1}k}) \;=\; g_{\boldsymbol{a}}{}^{\boldsymbol{a}'}g^a{}_{\boldsymbol{a}'}(\boldsymbol{k})g^{\boldsymbol{b}}{}_{\boldsymbol{b}'}\Lambda_a{}^b\tilde{g}_b{}^{\boldsymbol{b}'}(\boldsymbol{\Lambda^{-1}k}) \;=\; g_{\boldsymbol{a}}{}^{\boldsymbol{a}'}g^{\boldsymbol{b}}{}_{\boldsymbol{b}'}L_{\boldsymbol{a}'}{}^{\boldsymbol{b}'}(\Theta,\phi). \tag{307}$$

Recall that $g_{\boldsymbol{a}}{}^{\boldsymbol{a'}}$ and $g_{\boldsymbol{b'}}{}^{\boldsymbol{b}}$ are the Infeld-van der Waerden symbols introduced earlier in (36)–(37), the matrix $L_{\boldsymbol{a}'}{}^{\boldsymbol{b}'}$ links two null tetrads (30), and

$$\begin{aligned} L_{\boldsymbol{a}'}{}^{\boldsymbol{b}'}(\Theta,\phi) \;&=\; g^a{}_{\boldsymbol{a}'}(\boldsymbol{k})\Lambda_a{}^b\tilde{g}_b{}^{\boldsymbol{b}'}(\boldsymbol{\Lambda^{-1}k}) \;=\; \varepsilon_{\boldsymbol{A}}{}^A(\boldsymbol{k})\varepsilon_{\boldsymbol{A}'}{}^{A'}(\boldsymbol{k})\Lambda_A{}^B\Lambda_{A'}{}^{B'}\tilde{\varepsilon}_B{}^{\boldsymbol{B}}(\boldsymbol{\Lambda^{-1}k})\tilde{\varepsilon}_{B'}{}^{\boldsymbol{B}'}(\boldsymbol{\Lambda^{-1}k}) \\ &=\; L_{\boldsymbol{A}}{}^{\boldsymbol{B}}(\Theta,\phi)L_{\boldsymbol{A}'}{}^{\boldsymbol{B}'}(\Theta,\phi), \end{aligned} \tag{308}$$

where

$$L_{\boldsymbol{A}}{}^{\boldsymbol{B}}(\Theta,\phi) \;=\; \varepsilon_{\boldsymbol{A}}{}^A(\boldsymbol{k})\Lambda_A{}^B\tilde{\varepsilon}_B{}^{\boldsymbol{B}}(\boldsymbol{\Lambda^{-1}k}) \;=\; \varepsilon_{\boldsymbol{A}}{}^A(\boldsymbol{k})\Lambda\tilde{\varepsilon}_A{}^{\boldsymbol{B}}(\boldsymbol{k}) \tag{309}$$

is a SL(2,C) matrix which can explicitly be written in terms of spin-frames (21) and (22), i.e.

$$\begin{aligned} L_{\boldsymbol{A}}{}^{\boldsymbol{B}}(\Theta,\phi) &= \begin{pmatrix} L_{\boldsymbol{0}}{}^{\boldsymbol{0}}(\Theta,\phi) & L_{\boldsymbol{0}}{}^{\boldsymbol{1}}(\Theta,\phi) \\ L_{\boldsymbol{1}}{}^{\boldsymbol{0}}(\Theta,\phi) & L_{\boldsymbol{1}}{}^{\boldsymbol{1}}(\Theta,\phi) \end{pmatrix} \\ &= \begin{pmatrix} \omega_A(\boldsymbol{k})\Lambda\pi^A(\boldsymbol{k}) & \omega^A(\boldsymbol{k})\Lambda\tilde{\omega}_A(\boldsymbol{k}) \\ 0 & \pi^A(\boldsymbol{k})\Lambda\tilde{\omega}_A(\boldsymbol{k}) \end{pmatrix}. \end{aligned} \tag{310}$$

From the spin-frame condition (296) one can show that $\det\left(L_{\boldsymbol{A}}{}^{\boldsymbol{B}}(\Theta,\phi)\right) = 1$. Furthermore, the correspondence between the $L_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Theta,\phi)$ matrix and the two SL(2,C) matrices $L_{\boldsymbol{A}}{}^{\boldsymbol{B}}(\Theta,\phi)$, $L_{\boldsymbol{A'}}{}^{\boldsymbol{B}'}(\Theta,\phi)$ reads

$$L_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Theta,\phi) = g_{\boldsymbol{a}}{}^{\boldsymbol{a'}} L_{\boldsymbol{A}}{}^{\boldsymbol{B}}(\Theta,\phi) L_{\boldsymbol{A'}}{}^{\boldsymbol{B}'}(\Theta,\phi)\, g_{\boldsymbol{b'}}{}^{\boldsymbol{b}}. \tag{311}$$

5.3 Explicit transformation of tetrads

Now the transformation rule for the null tetrad can be written in the form

$$\begin{pmatrix} \Lambda\tilde{\omega}^a(\boldsymbol{k}) \\ \Lambda\tilde{m}^a(\boldsymbol{k}) \\ \Lambda\bar{\tilde{m}}^a(\boldsymbol{k}) \\ \Lambda k^a(\boldsymbol{k}) \end{pmatrix} = \begin{pmatrix} \Lambda\tilde{\omega}^A(\boldsymbol{k})\overline{\Lambda\tilde{\omega}}^{A'}(\boldsymbol{k}) \\ \Lambda\tilde{\omega}^A(\boldsymbol{k})\overline{\Lambda\pi}^{A'}(\boldsymbol{k}) \\ \Lambda\pi^A(\boldsymbol{k})\overline{\Lambda\tilde{\omega}}^{A'}(\boldsymbol{k}) \\ \Lambda\pi^A(\boldsymbol{k})\overline{\Lambda\pi}^{A'}(\boldsymbol{k}) \end{pmatrix} = \begin{pmatrix} \omega^a(\boldsymbol{k}) + \phi\; m^a(\boldsymbol{k}) + \bar{\phi}\; \bar{m}^a(\boldsymbol{k}) + |\phi|^2\; k^a(\boldsymbol{k}) \\ e^{2i\Theta}\; m^a(\boldsymbol{k}) + \phi e^{i\Theta}\; k^a(\boldsymbol{k}) \\ e^{-2i\Theta}\; \bar{m}^a(\boldsymbol{k}) + \bar{\phi} e^{-i\Theta}\; k^a(\boldsymbol{k}) \\ k^a(\boldsymbol{k}) \end{pmatrix} \tag{312}$$

Also the transformation rule for the Minkowski tetrad explicitly reads

$$\begin{pmatrix} \Lambda\tilde{t}^a(\boldsymbol{k}) \\ \Lambda\tilde{x}^a(\boldsymbol{k}) \\ \Lambda\tilde{y}^a(\boldsymbol{k}) \\ \Lambda\tilde{z}^a(\boldsymbol{k}) \end{pmatrix} = \begin{pmatrix} (1+|\phi|^2/2)\; t^a(\boldsymbol{k}) + |\phi|\cos\xi\; x^a(\boldsymbol{k}) - |\phi|\sin\xi\; y^a(\boldsymbol{k}) - |\phi|^2/2\; z^a(\boldsymbol{k}) \\ |\phi|\cos(2\Theta+\xi)\; t^a(\boldsymbol{k}) + \cos 2\Theta\; x^a(\boldsymbol{k}) + \sin 2\Theta\; y^a(\boldsymbol{k}) - |\phi|\cos(2\Theta+\xi)\; z^a(\boldsymbol{k}) \\ -|\phi|\sin(2\Theta+\xi)\; t^a(\boldsymbol{k}) - \sin 2\Theta\; x^a(\boldsymbol{k}) + \cos 2\Theta\; y^a(\boldsymbol{k}) + |\phi|\sin(2\Theta+\xi)\; z^a(\boldsymbol{k}) \\ |\phi|^2/2\; t^a(\boldsymbol{k}) + |\phi|\cos\xi\; x^a(\boldsymbol{k}) - |\phi|\sin\xi\; y^a(\boldsymbol{k}) + (1-|\phi|^2/2)\; z^a(\boldsymbol{k}) \end{pmatrix}. \tag{313}$$

The matrix (301) can be now written explicitly in terms of the Minkowski tetrad and the transformation parametrization from the spin-frame level, i.e. parameterized by $\Theta(\Lambda,\boldsymbol{k})$, $\phi(\boldsymbol{k})$.

$$\begin{aligned} L_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Theta,\phi) &= g^a{}_{\boldsymbol{a}}(\boldsymbol{k})\Lambda_a{}^b\tilde{g}_b{}^{\boldsymbol{b}}(\boldsymbol{\Lambda^{-1}k}) = g^a{}_{\boldsymbol{a}}(\boldsymbol{k})\Lambda\tilde{g}_a{}^{\boldsymbol{b}}(\boldsymbol{k}) && (314) \\ &= \begin{pmatrix} g^a{}_0(\boldsymbol{k})\Lambda\tilde{g}_a{}^0(\boldsymbol{k}) & g^a{}_0(\boldsymbol{k})\Lambda\tilde{g}_a{}^1(\boldsymbol{k}) & g^a{}_0(\boldsymbol{k})\Lambda\tilde{g}_a{}^2(\boldsymbol{k}) & g^a{}_0(\boldsymbol{k})\Lambda\tilde{g}_a{}^3(\boldsymbol{k}) \\ g^a{}_1(\boldsymbol{k})\Lambda\tilde{g}_a{}^0(\boldsymbol{k}) & g^a{}_1(\boldsymbol{k})\Lambda\tilde{g}_a{}^1(\boldsymbol{k}) & g^a{}_1(\boldsymbol{k})\Lambda\tilde{g}_a{}^2(\boldsymbol{k}) & g^a{}_1(\boldsymbol{k})\Lambda\tilde{g}_a{}^3(\boldsymbol{k}) \\ g^a{}_2(\boldsymbol{k})\Lambda\tilde{g}_a{}^0(\boldsymbol{k}) & g^a{}_2(\boldsymbol{k})\Lambda\tilde{g}_a{}^1(\boldsymbol{k}) & g^a{}_2(\boldsymbol{k})\Lambda\tilde{g}_a{}^2(\boldsymbol{k}) & g^a{}_2(\boldsymbol{k})\Lambda\tilde{g}_a{}^3(\boldsymbol{k}) \\ g^a{}_3(\boldsymbol{k})\Lambda\tilde{g}_a{}^0(\boldsymbol{k}) & g^a{}_3(\boldsymbol{k})\Lambda\tilde{g}_a{}^1(\boldsymbol{k}) & g^a{}_3(\boldsymbol{k})\Lambda\tilde{g}_a{}^2(\boldsymbol{k}) & g^a{}_3(\boldsymbol{k})\Lambda\tilde{g}_a{}^3(\boldsymbol{k}) \end{pmatrix} && (315) \\ &= \begin{pmatrix} t_a(\boldsymbol{k})\Lambda\tilde{t}^a(\boldsymbol{k}) & -t_a(\boldsymbol{k})\Lambda\tilde{x}^a(\boldsymbol{k}) & -t_a(\boldsymbol{k})\Lambda\tilde{y}^a(\boldsymbol{k}) & -t_a(\boldsymbol{k})\Lambda\tilde{z}^a(\boldsymbol{k}) \\ x_a(\boldsymbol{k})\Lambda\tilde{t}^a(\boldsymbol{k}) & -x_a(\boldsymbol{k})\Lambda\tilde{x}^a(\boldsymbol{k}) & -x_a(\boldsymbol{k})\Lambda\tilde{y}^a(\boldsymbol{k}) & -x_a(\boldsymbol{k})\Lambda\tilde{z}^a(\boldsymbol{k}) \\ y_a(\boldsymbol{k})\Lambda\tilde{t}^a(\boldsymbol{k}) & -y_a(\boldsymbol{k})\Lambda\tilde{x}^a(\boldsymbol{k}) & -y_a(\boldsymbol{k})\Lambda\tilde{y}^a(\boldsymbol{k}) & -y_a(\boldsymbol{k})\Lambda\tilde{z}^a(\boldsymbol{k}) \\ z_a(\boldsymbol{k})\Lambda\tilde{t}^a(\boldsymbol{k}) & -z_a(\boldsymbol{k})\Lambda\tilde{x}^a(\boldsymbol{k}) & -z_a(\boldsymbol{k})\Lambda\tilde{y}^a(\boldsymbol{k}) & -z_a(\boldsymbol{k})\Lambda\tilde{z}^a(\boldsymbol{k}) \end{pmatrix} && (316) \\ &= \begin{pmatrix} 1+\frac{|\phi|^2}{2} & -|\phi|\cos(\xi+2\Theta) & |\phi|\sin(\xi+2\Theta) & -\frac{|\phi|^2}{2} \\ -|\phi|\cos\xi & \cos 2\Theta & -\sin 2\Theta & |\phi|\cos\xi \\ |\phi|\sin\xi & \sin 2\Theta & \cos 2\Theta & -|\phi|\sin\xi \\ \frac{|\phi|^2}{2} & -|\phi|\cos(\xi+2\Theta) & |\phi|\sin(\xi+2\Theta) & 1-\frac{|\phi|^2}{2} \end{pmatrix}. && (317) \end{aligned}$$

Here, of course, $\Theta = \Theta(\Lambda,\boldsymbol{k})$, $|\phi| = |\phi(\boldsymbol{k})|$ and $\xi = \xi(\boldsymbol{k})$. More of the matrix index manipulation, i.e. lowering and raising indices by means of the metric tensor is shown in appendix E. Just for completeness, let us note that $L_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Theta,\phi)$ corresponds to the SL(2,C) matrix of the form

$$\begin{aligned} L_{\boldsymbol{A}}{}^{\boldsymbol{B}}(\Theta,\phi) &= \varepsilon_{\boldsymbol{A}}{}^A(\boldsymbol{k})\Lambda_A{}^B\tilde{\varepsilon}_B{}^{\boldsymbol{B}}(\boldsymbol{\Lambda^{-1}k}) = \varepsilon_{\boldsymbol{A}}{}^A(\boldsymbol{k})\Lambda\tilde{\varepsilon}_A{}^{\boldsymbol{B}}(\boldsymbol{k}) \\ &= \begin{pmatrix} \omega_A(\boldsymbol{k})\Lambda\pi^A(\boldsymbol{k}) & \omega^A(\boldsymbol{k})\Lambda\tilde{\omega}_A(\boldsymbol{k}) \\ 0 & \pi^A(\boldsymbol{k})\Lambda\tilde{\omega}_A(\boldsymbol{k}) \end{pmatrix} \\ &= \begin{pmatrix} e^{-i\Theta(\Lambda,\boldsymbol{k})} & -\phi(\boldsymbol{k})e^{i\Theta(\Lambda,\boldsymbol{k})} \\ 0 & e^{i\Theta(\Lambda,\boldsymbol{k})} \end{pmatrix}, \end{aligned} \tag{318}$$

and it can be split into two SL(2,C) matrices corresponding to gauge and homogeneous Lorentz transformations, i.e.

$$L_{\boldsymbol{A}}{}^{\boldsymbol{B}}(\Theta,\phi) \;=\; \begin{pmatrix} 1 & -\phi(\boldsymbol{k}) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-i\Theta(\Lambda,\boldsymbol{k})} & 0 \\ 0 & e^{i\Theta(\Lambda,\boldsymbol{k})} \end{pmatrix}. \tag{319}$$

Let us define those two SL(2,C) matrices corresponding to the gauge transformation and Wigner rotations respectively:

$$L_{\boldsymbol{A}}{}^{\boldsymbol{B}}(0,\phi(\boldsymbol{k})) \;=\; G_{\boldsymbol{A}}{}^{\boldsymbol{B}}(\phi(\boldsymbol{k})) \;=\; \varepsilon_{\boldsymbol{A}}{}^{A}(\boldsymbol{k})\tilde{\varepsilon}_{A}{}^{\boldsymbol{B}}(\boldsymbol{k}) \;=\; \begin{pmatrix} 1 & -\phi(\boldsymbol{k}) \\ 0 & 1 \end{pmatrix}, \tag{320}$$

$$L_{\boldsymbol{A}}{}^{\boldsymbol{B}}(\Theta(\Lambda,\boldsymbol{k}),0) \;=\; R_{\boldsymbol{A}}{}^{\boldsymbol{B}}(\Lambda,\boldsymbol{k}) \;=\; \varepsilon_{\boldsymbol{A}}{}^{A}(\boldsymbol{k})\Lambda\varepsilon_{A}{}^{\boldsymbol{B}}(\boldsymbol{k}) \;=\; \begin{pmatrix} e^{-i\Theta(\Lambda,\boldsymbol{k})} & 0 \\ 0 & e^{i\Theta(\Lambda,\boldsymbol{k})} \end{pmatrix}. \tag{321}$$

5.4 Bogoliubov type transformation

At this point only the irreducible representation of CCR will be considered. Let us first define new ladder operators as follows

$$b_{\boldsymbol{a}} \;=\; L_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Theta,\phi)a_{\boldsymbol{b}}: \tag{322}$$

$$b_0 \;=\; L_0{}^0(\Theta,\phi)a_0 + L_0{}^1(\Theta,\phi)a_1 + L_0{}^2(\Theta,\phi)a_2 + L_0{}^3(\Theta,\phi)a_3, \tag{323}$$

$$b_1 \;=\; L_1{}^0(\Theta,\phi)a_0 + L_1{}^1(\Theta,\phi)a_1 + L_1{}^2(\Theta,\phi)a_2 + L_1{}^3(\Theta,\phi)a_3, \tag{324}$$

$$b_2 \;=\; L_2{}^0(\Theta,\phi)a_0 + L_2{}^1(\Theta,\phi)a_1 + L_2{}^2(\Theta,\phi)a_2 + L_2{}^3(\Theta,\phi)a_3, \tag{325}$$

$$b_3 \;=\; L_3{}^0(\Theta,\phi)a_0 + L_3{}^1(\Theta,\phi)a_1 + L_3{}^2(\Theta,\phi)a_2 + L_3{}^3(\Theta,\phi)a_3. \tag{326}$$

These new operators are expressed as a combination of the old ones. Since transformation (301) has the property of leaving the metric invariant, i.e.

$$g_{\boldsymbol{ab}} \;=\; L_{\boldsymbol{a}}{}^{\boldsymbol{c}}(\Theta,\phi)L_{\boldsymbol{b}}{}^{\boldsymbol{d}}(\Theta,\phi)g_{\boldsymbol{cd}}, \tag{327}$$

the new operators satisfy the same CCR

$$[b_{\boldsymbol{a}}, b_{\boldsymbol{b}}^{\dagger}] \;=\; [L_{\boldsymbol{a}}{}^{\boldsymbol{c}}(\Theta,\phi)a_{\boldsymbol{c}}\;, L_{\boldsymbol{b}}{}^{\boldsymbol{d}}(\Theta,\phi)a_{\boldsymbol{d}}^{\dagger}] \tag{328}$$

$$\;=\; -L_{\boldsymbol{a}}{}^{\boldsymbol{c}}(\Theta,\phi)L_{\boldsymbol{b}}{}^{\boldsymbol{d}}(\Theta,\phi)g_{\boldsymbol{cd}} \;=\; -\,g_{\boldsymbol{ab}}. \tag{329}$$

Therefore, there must exist a unitary Bogoliubov-type transformation $U(\Theta,\phi)$ satisfying

$$b_{\boldsymbol{a}} \;=\; U(\Theta,\phi)^{\dagger}a_{\boldsymbol{a}}U(\Theta,\phi) \;=\; L_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Theta,\phi)a_{\boldsymbol{b}}. \tag{330}$$

In the next section an explicit representation of $U(\Theta,\phi)$ will be given.

5.5 Rotations and gauge transformation in four-dimensional polarization representation

In 1939 Wigner studied subgroups of the Lorentz group, whose transformations leave the four-momentum of a given free particle invariant [20]. The maximal subgroup of the Lorentz group, which leaves the four momentum invariant is called the little group. This implies that the little group governs the internal space-time symmetries of relativistic particles. Wigner showed in his paper that invariant space-time symmetries are dictated by O(3) like little groups in case of massive particles and by E(2) like little groups in case of massless ones. The application for photons has been discussed in many papers, among all [22]–[27]. It is also known that the Lorentz group is a very natural language for polarized light. There is no rest frame for massless particles, we will however choose the momentum in the z direction. In order to explicitly construct transformation $U(\Theta,\phi)$ in (330), let us first introduce the representation of the Lie algebra generators of rotations $J_{\boldsymbol{i}}$ around an i-th axis. This first will be done in canonical variables (109) introduced earlier in chapter 4, i.e.

$$J_{\boldsymbol{i}} \;=\; \varepsilon_{\boldsymbol{ijk}}q_{\boldsymbol{j}}p_{\boldsymbol{k}}. \tag{331}$$

As mentioned before these variables should not be mistaken with the position or momentum of the field. These are space and time-like variables that are used for a covariant structure of the four polarization degrees of freedom. Using formulas (119)–(120) we can explicitly write the generators of rotations in terms of the ladder operators

$$J_{\boldsymbol{i}} = -i\varepsilon_{\boldsymbol{ijk}} a_{\boldsymbol{j}}^{\dagger} a_{\boldsymbol{k}} : \tag{332}$$

$$J_1 = i(a_3^{\dagger}a_2 - a_2^{\dagger}a_3), \qquad J_2 = i(a_1^{\dagger}a_3 - a_3^{\dagger}a_1), \qquad J_3 = i(a_2^{\dagger}a_1 - a_1^{\dagger}a_2). \tag{333}$$

Also let us introduce boosts $K_{\boldsymbol{i}}$ along an i-th axis, first in terms of canonical variables (109)

$$K_{\boldsymbol{i}} = p_{\boldsymbol{i}} q_0 - p_0 q_{\boldsymbol{i}}. \tag{334}$$

Again using formulas (119) - (120) we explicitly write the generators of boosts in terms of the ladder operators

$$K_{\boldsymbol{i}} = i(a_{\boldsymbol{i}}^{\dagger}a_0 - a_0^{\dagger}a_{\boldsymbol{i}}) : \tag{335}$$

$$K_1 = i(a_1^{\dagger}a_0 - a_0^{\dagger}a_1), \qquad K_2 = i(a_2^{\dagger}a_0 - a_0^{\dagger}a_2), \qquad K_3 = i(a_3^{\dagger}a_0 - a_0^{\dagger}a_3). \tag{336}$$

Such a form of generators was introduced earlier in [12]. These generators satisfy the following commutation relations:

$$[J_{\boldsymbol{i}}, J_{\boldsymbol{j}}] = i\varepsilon_{\boldsymbol{ijk}} J_{\boldsymbol{k}}, \tag{337}$$

$$[K_{\boldsymbol{i}}, K_{\boldsymbol{j}}] = -i\varepsilon_{\boldsymbol{ijk}} J_{\boldsymbol{k}}, \tag{338}$$

$$[J_{\boldsymbol{i}}, K_{\boldsymbol{j}}] = i\varepsilon_{\boldsymbol{ijk}} K_{\boldsymbol{k}}. \tag{339}$$

Now the Bogoliubov-type transformation $U(\Theta, \phi)$ for ladder operators (330) can be written as

$$U(\Theta, \phi) = \exp(i\boldsymbol{\alpha} \cdot \boldsymbol{J} + i\boldsymbol{\beta} \cdot \boldsymbol{K}), \tag{340}$$

with

$$\alpha_1(\Theta, \phi) = -\frac{\Theta(\Lambda, \boldsymbol{k})}{\sin\Theta(\Lambda, \boldsymbol{k})} |\phi(\boldsymbol{k})| \sin(\xi(\boldsymbol{k}) + \Theta(\Lambda, \boldsymbol{k})), \tag{341}$$

$$\alpha_2(\Theta, \phi) = -\frac{\Theta(\Lambda, \boldsymbol{k})}{\sin\Theta(\Lambda, \boldsymbol{k})} |\phi(\boldsymbol{k})| \cos(\xi(\boldsymbol{k}) + \Theta(\Lambda, \boldsymbol{k})), \tag{342}$$

$$\alpha_3(\Theta) = -2\Theta(\Lambda, \boldsymbol{k}), \tag{343}$$

$$\beta_1(\Theta, \phi) = \frac{\Theta(\Lambda, \boldsymbol{k})}{\sin\Theta(\Lambda, \boldsymbol{k})} |\phi(\boldsymbol{k})| \cos(\xi(\boldsymbol{k}) + \Theta(\Lambda, \boldsymbol{k})), \tag{344}$$

$$\beta_2(\Theta, \phi) = -\frac{\Theta(\Lambda, \boldsymbol{k})}{\sin\Theta(\Lambda, \boldsymbol{k})} |\phi(\boldsymbol{k})| \sin(\xi(\boldsymbol{k}) + \Theta(\Lambda, \boldsymbol{k})), \tag{345}$$

$$\beta_3 = 0. \tag{346}$$

As mentioned before, Wigner in his paper [20] showed that the little group for massless particles moving along z axis is generated by the rotation generators around z axis J_3 and two other generators. As one can see here the parameter α_3 depends only on the Wigner phase. The two other generators are combinations of J_i and K_i and form a representation of an Euclidean group E(2), i.e.

$$L_1 = J_1 + K_2, \qquad L_2 = J_2 - K_1, \tag{347}$$

$$L_3 = J_3, \tag{348}$$

which satisfy the following commutation relations

$$[L_1, L_3] = -iL_2, \qquad [L_2, L_3] = iL_1, \qquad [L_1, L_2] = 0. \tag{349}$$

The physical variable associated with J_3 is the helicity degree of freedom of massless particles, but it was not clear what is the physical interpretation of generators L_1 and L_2. In 1971 Janner and Janssen [21] showed that those generators generate translations and are responsible for gauge transformations of the four potential. This will be discussed further in section 5.11.

Moreover, it should be stressed that only L_3 annihilates the vacuum states, since it is normally ordered, contrary to generators L_1 and L_2.

Now the transformation can be written as

$$U(\Theta,\phi) \;=\; \exp(i\alpha_1 L_1 + i\alpha_2 L_2 + i\alpha_3 L_3), \tag{350}$$

with parameters

$$\alpha_1(\Theta,\phi) \;=\; -\frac{\Theta(\Lambda,\boldsymbol{k})}{\sin\Theta(\Lambda,\boldsymbol{k})}|\phi(\boldsymbol{k})|\sin(\xi(\boldsymbol{k})+\Theta(\Lambda,\boldsymbol{k})), \tag{351}$$

$$\alpha_2(\Theta,\phi) \;=\; -\frac{\Theta(\Lambda,\boldsymbol{k})}{\sin\Theta(\Lambda,\boldsymbol{k})}|\phi(\boldsymbol{k})|\cos(\xi(\boldsymbol{k})+\Theta(\Lambda,\boldsymbol{k})), \tag{352}$$

$$\alpha_3(\Theta) \;=\; -2\Theta(\Lambda,\boldsymbol{k}). \tag{353}$$

Also the new creation and annihilation operators associated with the Minkowski tetrad and the $L_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Theta,\phi)$ matrix can be written explicitly in terms of $\phi(\boldsymbol{k})$ and the Wigner phase $\Theta(\Lambda,\boldsymbol{k})$

$$U(\Theta,\phi)^\dagger a_0 U(\Theta,\phi) \;=\; -|\phi|\cos(\xi+2\Theta)a_1 + |\phi|\sin(\xi+2\Theta)a_2 - \frac{|\phi|^2}{2}a_3 + \left(1+\frac{|\phi|^2}{2}\right)a_0, \tag{354}$$

$$U(\Theta,\phi)^\dagger a_1 U(\Theta,\phi) \;=\; \cos 2\Theta a_1 - \sin 2\Theta a_2 + |\phi|\cos\xi a_3 - |\phi|\cos\xi a_0, \tag{355}$$

$$U(\Theta,\phi)^\dagger a_2 U(\Theta,\phi) \;=\; \sin 2\Theta a_1 + \cos 2\Theta a_2 - |\phi|\sin\xi a_3 + |\phi|\sin\xi a_0, \tag{356}$$

$$U(\Theta,\phi)^\dagger a_3 U(\Theta,\phi) \;=\; -|\phi|\cos(\xi+2\Theta)a_1 + |\phi|\sin(\xi+2\Theta)a_2 + \left(1-\frac{|\phi|^2}{2}\right)a_3 + \frac{|\phi|^2}{2}a_0, \tag{357}$$

$$L_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Theta,\phi) \;=\; \begin{pmatrix} 1+\frac{|\phi(\boldsymbol{k})|^2}{2} & -|\phi(\boldsymbol{k})|\cos(\xi(\boldsymbol{k})+2\Theta(\Lambda,\boldsymbol{k})) & |\phi(\boldsymbol{k})|\sin(\xi(\boldsymbol{k})+2\Theta(\Lambda,\boldsymbol{k})) & -\frac{|\phi(\boldsymbol{k})|^2}{2} \\ -|\phi(\boldsymbol{k})|\cos\xi(\boldsymbol{k}) & \cos 2\Theta(\Lambda,\boldsymbol{k}) & -\sin 2\Theta(\Lambda,\boldsymbol{k}) & |\phi(\boldsymbol{k})|\cos\xi(\boldsymbol{k}) \\ |\phi(\boldsymbol{k})|\sin\xi(\boldsymbol{k}) & \sin 2\Theta(\Lambda,\boldsymbol{k}) & \cos 2\Theta(\Lambda,\boldsymbol{k}) & -|\phi(\boldsymbol{k})|\sin\xi(\boldsymbol{k}) \\ \frac{|\phi(\boldsymbol{k})|^2}{2} & -|\phi(\boldsymbol{k})|\cos(\xi(\boldsymbol{k})+2\Theta(\Lambda,\boldsymbol{k})) & |\phi(\boldsymbol{k})|\sin(\xi(\boldsymbol{k})+2\Theta(\Lambda,\boldsymbol{k})) & 1-\frac{|\phi(\boldsymbol{k})|^2}{2} \end{pmatrix}, \tag{358}$$

$$U(\Theta,\phi) \;=\; \exp\left(-i\frac{\Theta|\phi|\sin(\xi+\Theta)}{\sin\Theta}L_1 - i\frac{\Theta|\phi|\cos(\xi+\Theta)}{\sin\Theta}L_2 - i2\Theta L_3\right). \tag{359}$$

For the inverse transformation we can write

$$U(\Theta,\phi)a_0 U(\Theta,\phi)^\dagger \;=\; |\phi|\cos\xi a_1 - |\phi|\sin\xi a_2 - \frac{|\phi|^2}{2}a_3 + \left(1+\frac{|\phi|^2}{2}\right)a_0, \tag{360}$$

$$U(\Theta,\phi)a_1 U(\Theta,\phi)^\dagger \;=\; \cos 2\Theta a_1 + \sin 2\Theta a_2 - |\phi|\cos(\xi+2\Theta)a_3 + |\phi|\cos(\xi+2\Theta)a_0, \tag{361}$$

$$U(\Theta,\phi)a_2 U(\Theta,\phi)^\dagger \;=\; -\sin 2\Theta a_1 + \cos 2\Theta a_2 + |\phi|\sin(\xi+2\Theta)a_3 - |\phi|\sin(\xi+2\Theta)a_0, \tag{362}$$

$$U(\Theta,\phi)a_3 U(\Theta,\phi)^\dagger \;=\; |\phi|\cos\xi a_1 - |\phi|\sin\xi a_2 + \left(1-\frac{|\phi|^2}{2}\right)a_3 + \frac{|\phi|^2}{2}a_0, \tag{363}$$

and from (306) we can calculate the matrix associated with the inverse transformation:

$$L^{-1}{}_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Theta,\phi) \;=\; \begin{pmatrix} 1+\frac{|\phi(\boldsymbol{k})|^2}{2} & |\phi(\boldsymbol{k})|\cos\xi(\boldsymbol{k}) & -|\phi(\boldsymbol{k})|\sin\xi(\boldsymbol{k}) & -\frac{|\phi(\boldsymbol{k})|^2}{2} \\ |\phi(\boldsymbol{k})|\cos(\xi(\boldsymbol{k})+2\Theta(\Lambda,\boldsymbol{k})) & \cos 2\Theta(\Lambda,\boldsymbol{k}) & \sin 2\Theta(\Lambda,\boldsymbol{k}) & -|\phi(\boldsymbol{k})|\cos(\xi(\boldsymbol{k})+2\Theta(\Lambda,\boldsymbol{k})) \\ -|\phi(\boldsymbol{k})|\sin(\xi(\boldsymbol{k})+2\Theta(\Lambda,\boldsymbol{k})) & -\sin 2\Theta(\Lambda,\boldsymbol{k}) & \cos 2\Theta(\Lambda,\boldsymbol{k}) & |\phi(\boldsymbol{k})|\sin(\xi(\boldsymbol{k})+2\Theta(\Lambda,\boldsymbol{k})) \\ \frac{|\phi(\boldsymbol{k})|^2}{2} & |\phi(\boldsymbol{k})|\cos\xi(\boldsymbol{k}) & -|\phi(\boldsymbol{k})|\sin\xi(\boldsymbol{k}) & 1-\frac{|\phi(\boldsymbol{k})|^2}{2} \end{pmatrix}. \tag{364}$$

To see how the $L_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Theta,\phi)$ transformation acts on operators associated with the null tetrad, we can calculate

$$U(\Theta,\phi)^\dagger a_{\boldsymbol{a}'} U(\Theta,\phi) \;=\; U(\Theta,\phi)^\dagger g_{\boldsymbol{a}'}{}^{\boldsymbol{a}} a_{\boldsymbol{a}} U(\Theta,\phi) \;=\; g_{\boldsymbol{a}'}{}^{\boldsymbol{a}} L_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Theta,\phi) g_{\boldsymbol{b}}{}^{\boldsymbol{b}'} a_{\boldsymbol{b}'} \;=\; L_{\boldsymbol{a}'}{}^{\boldsymbol{b}'}(\Theta,\phi) a_{\boldsymbol{b}'}, \tag{365}$$

where the transformation matrix

$$L_{\boldsymbol{a}'}{}^{\boldsymbol{b}'}(\Theta,\phi) \;=\; g_{\boldsymbol{a}'}{}^{\boldsymbol{a}} L_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Theta,\phi) g_{\boldsymbol{b}}{}^{\boldsymbol{b}'} \tag{366}$$

takes the explicit form

$$L_{\boldsymbol{a}'}{}^{\boldsymbol{b}'}(\Theta,\phi) \;=\; \begin{pmatrix} 1 & -|\phi(\boldsymbol{k})|e^{-i(\xi(\boldsymbol{k})+2\Theta(\Lambda,\boldsymbol{k}))} & -|\phi(\boldsymbol{k})|e^{i(\xi(\boldsymbol{k})+2\Theta(\Lambda,\boldsymbol{k}))} & |\phi(\boldsymbol{k})|^2 \\ 0 & e^{-2i\Theta(\Lambda,\boldsymbol{k})} & 0 & -|\phi(\boldsymbol{k})|e^{i\xi(\boldsymbol{k})} \\ 0 & 0 & e^{2i\Theta(\Lambda,\boldsymbol{k})} & -|\phi(\boldsymbol{k})|e^{-i\xi(\boldsymbol{k})} \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{367}$$

We can also write formula (359) in the following useful form

$$\begin{aligned} U(\Theta,\phi) &= \exp(i\alpha_1 L_1 + i\alpha_2 L_2 + i\alpha_3 L_3) \\ &= \exp\left(-i|\phi|\sin(\xi+2\Theta)L_1 - i|\phi|\cos(\xi+2\Theta)L_2\right)\exp\left(-2i\Theta L_3\right) &(368) \\ &= \exp\left(-2i\Theta L_3\right)\exp\left(-i|\phi|\sin\xi L_1 - i|\phi|\cos\xi L_2\right). &(369) \end{aligned}$$

The last formula (369) indicates that the general transformation matrix $L_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Theta,\phi)$ can be written as a product of the rotation matrix $R_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Lambda,\boldsymbol{k})$ and a gauge transformation matrix $G_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\boldsymbol{k})$, i.e.

$$\begin{aligned} L_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Theta,\phi) &= G_{\boldsymbol{a}}{}^{\boldsymbol{c}}(\boldsymbol{k}) R_{\boldsymbol{c}}{}^{\boldsymbol{b}}(\Lambda,\boldsymbol{k}) \\ = &\begin{pmatrix} 1+\frac{|\phi(\boldsymbol{k})|^2}{2} & -|\phi(\boldsymbol{k})|\cos\xi(\boldsymbol{k}) & |\phi(\boldsymbol{k})|\sin\xi(\boldsymbol{k}) & -\frac{|\phi(\boldsymbol{k})|^2}{2} \\ -|\phi(\boldsymbol{k})|\cos\xi(\boldsymbol{k}) & 1 & 0 & |\phi(\boldsymbol{k})|\cos\xi(\boldsymbol{k}) \\ |\phi(\boldsymbol{k})|\sin\xi(\boldsymbol{k}) & 0 & 1 & -|\phi(\boldsymbol{k})|\sin\xi(\boldsymbol{k}) \\ \frac{|\phi(\boldsymbol{k})|^2}{2} & -|\phi(\boldsymbol{k})|\cos\xi(\boldsymbol{k}) & |\phi(\boldsymbol{k})|\sin\xi(\boldsymbol{k}) & 1-\frac{|\phi(\boldsymbol{k})|^2}{2} \end{pmatrix} \\ \times &\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\Theta(\Lambda,\boldsymbol{k}) & -\sin 2\Theta(\Lambda,\boldsymbol{k}) & 0 \\ 0 & \sin 2\Theta(\Lambda,\boldsymbol{k}) & \cos 2\Theta(\Lambda,\boldsymbol{k}) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{370}$$

Of course, those two matrices do not commute, but it can be shown, from (368) and (299), that

$$L_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Theta,\phi) \;=\; G_{\boldsymbol{a}}{}^{\boldsymbol{c}}(\boldsymbol{k}) R_{\boldsymbol{c}}{}^{\boldsymbol{b}}(\Lambda,\boldsymbol{k}) \;=\; R_{\boldsymbol{a}}{}^{\boldsymbol{c}}(\Lambda,\boldsymbol{k}) G_{\boldsymbol{c}}{}^{\boldsymbol{b}}(\boldsymbol{\Lambda^{-1}k}). \tag{371}$$

For further purposes we will define transformations associated with the Lorentz transformation and gauge transformation respectively

$$\begin{aligned} U(\Lambda,\boldsymbol{k}) &= \exp\left(-2i\Theta(\Lambda,\boldsymbol{k})L_3\right), &(372) \\ U_G(\boldsymbol{k}) &= \exp\left(-i|\phi(\boldsymbol{k})|\sin\xi(\boldsymbol{k})L_1 - i|\phi(\boldsymbol{k})|\cos\xi(\boldsymbol{k})L_2\right), &(373) \end{aligned}$$

such that

$$\begin{aligned} U(\Lambda,\boldsymbol{k})^\dagger a_{\boldsymbol{a}} U(\Lambda,\boldsymbol{k}) &= R_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Lambda,\boldsymbol{k}) a_{\boldsymbol{b}}, \\ U_G(\boldsymbol{k})^\dagger a_{\boldsymbol{a}} U_G(\boldsymbol{k}) &= G_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\boldsymbol{k}) a_{\boldsymbol{b}}. &(374) \end{aligned}$$

5.6 Lorentz transformation for the reducible representation

To construct a Lorentz transformation for the reducible $N=1$ oscillator representation the Bogoliubov-type transformation must be written as

$$U(\Lambda,0,1) \;=\; \int d\Gamma(\boldsymbol{k})|\boldsymbol{k}\rangle\langle\boldsymbol{\Lambda^{-1}k}| \otimes U(\Lambda,\boldsymbol{k}). \tag{375}$$

Then for the hermitian conjugate we can write

$$U(\Lambda,0,1)^\dagger \;=\; \int d\Gamma(\boldsymbol{k})|\boldsymbol{\Lambda^{-1}k}\rangle\langle\boldsymbol{k}| \otimes U(\Lambda,\boldsymbol{k})^\dagger. \tag{376}$$

Let us denote

$$W(\Lambda) \;=\; \int d\Gamma(\boldsymbol{k})|\boldsymbol{k}\rangle\langle\boldsymbol{\Lambda^{-1}k}|. \tag{377}$$

This operator is not dependent on spin and acts only on momentum $\boldsymbol{k}$, i.e.

$$W(\Lambda)|\boldsymbol{k}\rangle \;=\; |\boldsymbol{\Lambda k}\rangle. \tag{378}$$

Operator (377) leaves the inner product invariant and therefore is an unitary operator. The hermitian conjugate of (377) is

$$\begin{aligned} W(\Lambda)^{\dagger} &= \int d\Gamma(\boldsymbol{k})|\boldsymbol{\Lambda^{-1}k}\rangle\langle\boldsymbol{k}|, && (379)\\ W(\Lambda)^{\dagger} &= W(\Lambda^{-1}), && (380)\\ W(\Lambda)^{\dagger}|\boldsymbol{k}\rangle &= |\boldsymbol{\Lambda^{-1}k}\rangle. && (381)\end{aligned}$$

Furthermore, operators (377) and (379) have the following properties:

$$\begin{aligned} W(\Lambda)W(\Lambda)^{\dagger} &= W(\Lambda)W(\Lambda^{-1}) \;=\; I, && (382)\\ W(\Lambda)^{\dagger}\;|\boldsymbol{k}\rangle\langle\boldsymbol{k}|\;W(\Lambda) &= |\boldsymbol{\Lambda^{-1}k}\rangle\langle\boldsymbol{\Lambda^{-1}k}|, && (383)\\ W(\Lambda)\;|\boldsymbol{k}\rangle\langle\boldsymbol{k}|\;W(\Lambda)^{\dagger} &= |\boldsymbol{\Lambda k}\rangle\langle\boldsymbol{\Lambda k}|. && (384)\end{aligned}$$

For the N-oscillator extension we can write

$$U(\Lambda,0,N) \;=\; U(\Lambda,0,1)^{\otimes N}. \tag{385}$$

Then, for circular polarizations under Lorentz transformation we get the following transformation rules for creation and annihilation operators in N-oscillator reducible representations

$$\begin{aligned} U(\Lambda,0,N)^{\dagger}a_s(\boldsymbol{k},N)U(\Lambda,0,N) &= e^{-2is\Theta(\Lambda,\boldsymbol{k})}a_s(\boldsymbol{\Lambda^{-1}k},N), && (386)\\ U(\Lambda,0,N)^{\dagger}a_s(\boldsymbol{k},N)^{\dagger}U(\Lambda,0,N) &= e^{2is\Theta(\Lambda,\boldsymbol{k})}a_s(\boldsymbol{\Lambda^{-1}k},N)^{\dagger}, && (387)\\ U(\Lambda,0,N)a_s(\boldsymbol{k},N)U(\Lambda,0,N)^{\dagger} &= e^{2is\Theta(\Lambda,\boldsymbol{\Lambda k})}a_s(\boldsymbol{\Lambda k},N), && (388)\\ U(\Lambda,0,N)a_s(\boldsymbol{k},N)^{\dagger}U(\Lambda,0,N)^{\dagger} &= e^{-2is\Theta(\Lambda,\boldsymbol{\Lambda k})}a_s(\boldsymbol{\Lambda k},N)^{\dagger}. && (389)\end{aligned}$$

5.7 Composition law

From (377) and (379) it is easy to show that $W(\Lambda)$ and $W(\Lambda)^{\dagger}$ satisfy the following composition law:

$$\begin{aligned} W(\Lambda)W(\Lambda')|\boldsymbol{k}\rangle &= |\boldsymbol{\Lambda\Lambda' k}\rangle \;=\; W(\Lambda\Lambda')|\boldsymbol{k}\rangle,\\ W(\Lambda)^{\dagger}W(\Lambda')^{\dagger}|\boldsymbol{k}\rangle &= |\boldsymbol{\Lambda^{-1}\Lambda'^{-1}k}\rangle \;=\; |\boldsymbol{(\Lambda'\Lambda)^{-1}k}\rangle \;=\; W(\Lambda'\Lambda)^{\dagger}|\boldsymbol{k}\rangle. && (390)\end{aligned}$$

This means that they are unitary representations of the Lorentz group. We shall now take a closer look at the composition law for the reducible representations of $U(\Lambda,0,1)$ (375):

$$\begin{aligned} U(\Lambda,0,1)U(\Lambda',0,1) &= \Big(\int d\Gamma(\boldsymbol{k})|\boldsymbol{k}\rangle\langle\boldsymbol{\Lambda^{-1}k}|\otimes U(\Lambda,\boldsymbol{k})\Big)\Big(\int d\Gamma(\boldsymbol{k}')|\boldsymbol{k}'\rangle\langle\boldsymbol{\Lambda'^{-1}k'}|\otimes U(\Lambda',\boldsymbol{k}')\Big)\\ &= \int d\Gamma(\boldsymbol{k})\int d\Gamma(\boldsymbol{k}')|\boldsymbol{k}\rangle\langle\boldsymbol{\Lambda'^{-1}k'}|\delta_\Gamma(\boldsymbol{\Lambda^{-1}k},\boldsymbol{k}')\otimes U(\Lambda,\boldsymbol{k})U(\Lambda',\boldsymbol{k}')\\ &= \int d\Gamma(\boldsymbol{k})|\boldsymbol{k}\rangle\langle\boldsymbol{(\Lambda\Lambda')^{-1}k}|\otimes U(\Lambda,\boldsymbol{k})U(\Lambda',\boldsymbol{\Lambda^{-1}k}). && (391)\end{aligned}$$

The left-hand side of the above should be equal to $\int d\Gamma(\boldsymbol{k})|\boldsymbol{k}\rangle\langle\boldsymbol{(\Lambda\Lambda')^{-1}k}|\otimes U(\Lambda\Lambda',\boldsymbol{k})$ and this implies the following condition

$$U(\Lambda\Lambda',\boldsymbol{k}) \;=\; U(\Lambda,\boldsymbol{k})U(\Lambda',\boldsymbol{\Lambda^{-1}k}). \tag{392}$$

For the hermitian conjugates we can write:

$$\begin{aligned}
U(\Lambda',0,1)^\dagger U(\Lambda,0,1)^\dagger &= \left(\int d\Gamma(\boldsymbol{k}')|\boldsymbol{\Lambda}'^{-1}\boldsymbol{k}'\rangle\langle\boldsymbol{k}'|\otimes U(\Lambda',\boldsymbol{k}')^\dagger\right)\left(\int d\Gamma(\boldsymbol{k})|\boldsymbol{\Lambda}^{-1}\boldsymbol{k}\rangle\langle\boldsymbol{k}|\otimes U(\Lambda,\boldsymbol{k})^\dagger\right)\\
&= \int d\Gamma(\boldsymbol{k})\int d\Gamma(\boldsymbol{k}')|\boldsymbol{\Lambda}'^{-1}\boldsymbol{k}'\rangle\langle\boldsymbol{k}|\delta_\Gamma(\boldsymbol{k}',\boldsymbol{\Lambda}^{-1}\boldsymbol{k})\otimes U(\Lambda',\boldsymbol{k}')^\dagger U(\Lambda,\boldsymbol{k})^\dagger\\
&= \int d\Gamma(\boldsymbol{k})|(\boldsymbol{\Lambda}\boldsymbol{\Lambda}')^{-1}\boldsymbol{k}\rangle\langle\boldsymbol{k}|\otimes U(\Lambda',\boldsymbol{\Lambda}^{-1}\boldsymbol{k})^\dagger U(\Lambda,\boldsymbol{k})^\dagger. \qquad (393)
\end{aligned}$$

This implies

$$U(\Lambda\Lambda',\boldsymbol{k})^\dagger \;=\; U(\Lambda',\boldsymbol{\Lambda}^{-1}\boldsymbol{k})^\dagger U(\Lambda,\boldsymbol{k})^\dagger. \qquad (394)$$

The reducible representation is unitary thus

$$\begin{aligned}
U(\Lambda,0,1)U(\Lambda,0,1)^\dagger &= \left(\int d\Gamma(\boldsymbol{k})|\boldsymbol{k}\rangle\langle\boldsymbol{\Lambda}^{-1}\boldsymbol{k}|\otimes U(\Lambda,\boldsymbol{k})\right)\left(\int d\Gamma(\boldsymbol{k}')|\boldsymbol{\Lambda}^{-1}\boldsymbol{k}'\rangle\langle\boldsymbol{k}'|\otimes U(\Lambda,\boldsymbol{k}')^\dagger\right)\\
&= \int d\Gamma(\boldsymbol{k})\int d\Gamma(\boldsymbol{k}')|\boldsymbol{k}\rangle\langle\boldsymbol{k}'|\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}')\otimes U(\Lambda,\boldsymbol{k})U(\Lambda,\boldsymbol{k}')^\dagger\\
&= \int d\Gamma(\boldsymbol{k})|\boldsymbol{k}\rangle\langle\boldsymbol{k}|\otimes U(\Lambda,\boldsymbol{k})U(\Lambda,\boldsymbol{k})^\dagger\\
&= I\otimes 1_4. \qquad (395)
\end{aligned}$$

On the other hand from (391) putting $\Lambda'=\Lambda^{-1}$ we get

$$\begin{aligned}
U(\Lambda,0,1)U(\Lambda^{-1},0,1) &= \left(\int d\Gamma(\boldsymbol{k})|\boldsymbol{k}\rangle\langle\boldsymbol{\Lambda}^{-1}\boldsymbol{k}|\otimes U(\Lambda,\boldsymbol{k})\right)\left(\int d\Gamma(\boldsymbol{k}')|\boldsymbol{k}'\rangle\langle\boldsymbol{\Lambda}\boldsymbol{k}'|\otimes U(\Lambda^{-1},\boldsymbol{k}')\right)\\
&= \int d\Gamma(\boldsymbol{k})\int d\Gamma(\boldsymbol{k}')|\boldsymbol{k}\rangle\langle\boldsymbol{\Lambda}\boldsymbol{k}'|\delta_\Gamma(\boldsymbol{\Lambda}^{-1}\boldsymbol{k},\boldsymbol{k}')\otimes U(\Lambda,\boldsymbol{k})U(\Lambda^{-1},\boldsymbol{k}')\\
&= \int d\Gamma(\boldsymbol{k})|\boldsymbol{k}\rangle\langle\boldsymbol{k}|\otimes U(\Lambda,\boldsymbol{k})U(\Lambda^{-1},\boldsymbol{\Lambda}^{-1}\boldsymbol{k})\\
&= I\otimes 1_4, \qquad (396)
\end{aligned}$$

and this implies

$$U(\Lambda,\boldsymbol{k})^\dagger \;=\; U(\Lambda^{-1},\boldsymbol{\Lambda}^{-1}\boldsymbol{k}). \qquad (397)$$

Condition (392) imposes a composition law for the $R_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Lambda,\boldsymbol{k})$ matrix such that

$$R_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Lambda\Lambda',\boldsymbol{k}) \;=\; R_{\boldsymbol{a}}{}^{\boldsymbol{c}}(\Lambda,\boldsymbol{k})R_{\boldsymbol{c}}{}^{\boldsymbol{b}}(\Lambda',\boldsymbol{\Lambda}^{-1}\boldsymbol{k}). \qquad (398)$$

The composition law will be shown here on the spinor level of SL(2,C) matrix. Let us first remind ourselves of formula (309)

$$R_{\boldsymbol{A}}{}^{\boldsymbol{B}}(\Lambda,\boldsymbol{k}) \;=\; \varepsilon_{\boldsymbol{A}}{}^{A}(\boldsymbol{k})\Lambda\varepsilon_{A}{}^{\boldsymbol{B}}(\boldsymbol{k}).$$

It is easy to show, using formula derived in the appendix (B.14), that

$$\begin{aligned}
R_{\boldsymbol{A}}{}^{\boldsymbol{C}}(\Lambda,\boldsymbol{k})R_{\boldsymbol{C}}{}^{\boldsymbol{B}}(\Lambda',\boldsymbol{\Lambda}^{-1}\boldsymbol{k}) &= \varepsilon_{\boldsymbol{A}}{}^{B}(\boldsymbol{k})\Lambda\varepsilon_{B}{}^{\boldsymbol{C}}(\boldsymbol{k})\varepsilon_{\boldsymbol{C}}{}^{A}(\boldsymbol{\Lambda}^{-1}\boldsymbol{k})\Lambda'\varepsilon_{A}{}^{\boldsymbol{B}}(\boldsymbol{\Lambda}^{-1}\boldsymbol{k})\\
&= \varepsilon_{\boldsymbol{A}}{}^{B}(\boldsymbol{k})\Lambda\varepsilon_{B}{}^{\boldsymbol{C}}(\boldsymbol{k})\Lambda\varepsilon_{\boldsymbol{C}}{}^{A}(\boldsymbol{k})\Lambda\Lambda'\varepsilon_{A}{}^{\boldsymbol{B}}(\boldsymbol{k})\\
&= \varepsilon_{\boldsymbol{A}}{}^{B}(\boldsymbol{k})\varepsilon_{B}{}^{A}\Lambda\Lambda'\varepsilon_{A}{}^{\boldsymbol{B}}(\boldsymbol{k})\\
&= \varepsilon_{\boldsymbol{A}}{}^{A}(\boldsymbol{k})\Lambda\Lambda'\varepsilon_{A}{}^{\boldsymbol{B}}(\boldsymbol{k}) \;=\; R_{\boldsymbol{A}}{}^{\boldsymbol{B}}(\Lambda\Lambda',\boldsymbol{k}). \qquad (399)
\end{aligned}$$

From this, and the explicit value of the matrix $R_{\boldsymbol{A}}{}^{\boldsymbol{C}}(\Lambda,\boldsymbol{k})$ (321), it can be shown that

$$e^{i\Theta(\Lambda,\boldsymbol{k})}e^{i\Theta(\Lambda',\boldsymbol{\Lambda}^{-1}\boldsymbol{k})} \;=\; e^{i\Theta(\Lambda\Lambda',\boldsymbol{k})}. \qquad (400)$$

This is derived explicitly in appendix (B.20). Furthermore, let us also prove the composition law of Lorentz transformations on the gauge parameter $\phi(\boldsymbol{k})$, i.e.

$$\Lambda\Lambda'\phi(\boldsymbol{k}) \;=\; \phi((\boldsymbol{\Lambda}\boldsymbol{\Lambda}')^{-1}\boldsymbol{k}) \;=\; e^{2i\Theta(\Lambda\Lambda',\boldsymbol{k})}\phi(\boldsymbol{k}). \qquad (401)$$

Proof: Let

$$\Lambda\Lambda'\phi(\boldsymbol{k}) \;=\; \Lambda'\phi(\boldsymbol{\Lambda^{-1}k}) \;=\; \phi(\boldsymbol{\Lambda'^{-1}(\Lambda^{-1}k)}) \;=\; \phi(\boldsymbol{(\Lambda\Lambda')^{-1}k}). \tag{402}$$

Using such a notation and transformation rule (299) for $\phi(\boldsymbol{k})$, we can show that

$$\begin{aligned}
\phi(\boldsymbol{(\Lambda\Lambda')^{-1}k}) &= \Lambda'\phi(\boldsymbol{\Lambda^{-1}k}) \;=\; e^{2i\Theta(\Lambda',\boldsymbol{\Lambda^{-1}k})}\phi(\boldsymbol{\Lambda^{-1}k}) && (403)\\
&= e^{2i\Theta(\Lambda',\boldsymbol{\Lambda^{-1}k})}e^{2i\Theta(\Lambda,\boldsymbol{k})}\phi(\boldsymbol{k}) \;=\; e^{2i\Theta(\Lambda\Lambda',\boldsymbol{k})}\phi(\boldsymbol{k}). && (404)
\end{aligned}$$

5.8 Transformation properties of the potential operator

Now let us show that the potential operator transforms under Lorentz transformation as a four-vector, first for $N = 1$ oscillator representation.

$$\begin{aligned}
& U(\Lambda,0,1)^\dagger A_a(x,1)U(\Lambda,0,1)\\
&= \left(\int d\Gamma(\boldsymbol{l})|\boldsymbol{\Lambda^{-1}l}\rangle\langle\boldsymbol{l}| \otimes U(\Lambda,\boldsymbol{l})^\dagger\right)\left(i\int d\Gamma(\boldsymbol{k})g_a{}^{\boldsymbol{a}}(\boldsymbol{k})a_{\boldsymbol{b}}(\boldsymbol{k},1)e^{-ik\cdot x} + \text{H.c.}\right)\left(\int d\Gamma(\boldsymbol{l'})|\boldsymbol{l'}\rangle\langle\boldsymbol{\Lambda^{-1}l'}| \otimes U(\Lambda,\boldsymbol{l'})\right)\\
&= i\int d\Gamma(\boldsymbol{k})|\boldsymbol{\Lambda^{-1}k}\rangle\langle\boldsymbol{\Lambda^{-1}k}| \otimes g_a{}^{\boldsymbol{a}}(\boldsymbol{k})U(\Lambda,\boldsymbol{k})^\dagger a_{\boldsymbol{a}}U(\Lambda,\boldsymbol{k})e^{-ik\cdot x} + \text{H.c.}\\
&= i\int d\Gamma(\boldsymbol{k})|\boldsymbol{\Lambda^{-1}k}\rangle\langle\boldsymbol{\Lambda^{-1}k}| \otimes g_a{}^{\boldsymbol{a}}(\boldsymbol{k})R_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Lambda,\boldsymbol{k})a_{\boldsymbol{b}}e^{-ik\cdot x} + \text{H.c.}\\
&= i\int d\Gamma(\boldsymbol{k})\Lambda g_a{}^{\boldsymbol{b}}(\boldsymbol{k})a_{\boldsymbol{b}}(\boldsymbol{\Lambda^{-1}k},1)e^{-ik\cdot x} + \text{H.c.}\\
&= i\int d\Gamma(\boldsymbol{k})\Lambda_a{}^b g_b{}^{\boldsymbol{b}}(\boldsymbol{k})a_{\boldsymbol{b}}(\boldsymbol{k},1)e^{-ik\cdot\Lambda^{-1}x} + \text{H.c.}\\
&= \Lambda_a{}^b A_b(\Lambda^{-1}x,1).
\end{aligned} \tag{405}$$

The main element of the construction could be formulated at a level independent of properties of concrete representations of CCR. We can also extend this calculation to any natural number N, where the four-potential in N-oscillator representation is denoted by $A_a(x,N)$, i.e.

$$U(\Lambda,0,N)^\dagger A_a(x,N)U(\Lambda,0,N) \;=\; \Lambda_a{}^b A_b(\Lambda^{-1}x,N). \tag{406}$$

5.9 Transformation properties of electromagnetic field operator

Recall the electromagnetic field operator (251)

$$F_{ab}(x,1) \;=\; \int d\Gamma(\boldsymbol{k})\left(k_a(\boldsymbol{k})g_b{}^{\boldsymbol{a}}(\boldsymbol{k}) - k_b(\boldsymbol{k})g_a{}^{\boldsymbol{a}}(\boldsymbol{k})\right)a_{\boldsymbol{a}}(\boldsymbol{k},1)e^{-ik\cdot x} + \text{H.c.}$$

It can be shown that under Lorentz transformation the electromagnetic field operator transforms like a tensor:

$$
\begin{aligned}
& U(\Lambda,0,1)^\dagger F_{ab}(x,1)U(\Lambda,0,1) \\
= & \left(\int d\Gamma(\boldsymbol{l})|\boldsymbol{\Lambda^{-1}l}\rangle\langle\boldsymbol{l}|\otimes U(\Lambda,\boldsymbol{l})^\dagger\right) \\
\times & \left(\int d\Gamma(\boldsymbol{k})\left(k_a(\boldsymbol{k})g_b{}^{\boldsymbol{a}}(\boldsymbol{k})-k_b(\boldsymbol{k})g_a{}^{\boldsymbol{a}}(\boldsymbol{k})\right)a_{\boldsymbol{a}}(\boldsymbol{k},1)e^{-ik\cdot x}+\text{H.c.}\right) \\
\times & \left(\int d\Gamma(\boldsymbol{l'})|\boldsymbol{l'}\rangle\langle\boldsymbol{\Lambda^{-1}l'}|\otimes U(\Lambda,\boldsymbol{l'})\right) \\
= & \int d\Gamma(\boldsymbol{k})|\boldsymbol{\Lambda^{-1}k}\rangle\langle\boldsymbol{\Lambda^{-1}k}|\otimes\left(k_a(\boldsymbol{k})g_b{}^{\boldsymbol{a}}(\boldsymbol{k})-k_b(\boldsymbol{k})g_a{}^{\boldsymbol{a}}(\boldsymbol{k})\right)U(\Lambda,\boldsymbol{l'})^\dagger a_{\boldsymbol{a}}U(\Lambda,\boldsymbol{l'})e^{-ik\cdot x}+\text{H.c.} \\
= & \int d\Gamma(\boldsymbol{k})|\boldsymbol{\Lambda^{-1}k}\rangle\langle\boldsymbol{\Lambda^{-1}k}|\otimes\left(k_a(\boldsymbol{k})g_b{}^{\boldsymbol{a}}(\boldsymbol{k})-k_b(\boldsymbol{k})g_a{}^{\boldsymbol{a}}(\boldsymbol{k})\right)R_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Lambda,\boldsymbol{k})a_{\boldsymbol{b}}e^{-ik\cdot x}+\text{H.c.} \\
= & \int d\Gamma(\boldsymbol{k})|\boldsymbol{\Lambda^{-1}k}\rangle\langle\boldsymbol{\Lambda^{-1}k}|\otimes\left(k_a(\boldsymbol{k})g_b{}^{\boldsymbol{a}}(\boldsymbol{k})-k_b(\boldsymbol{k})g_a{}^{\boldsymbol{a}}(\boldsymbol{k})\right)g_{\boldsymbol{a}}{}^{c}(\boldsymbol{k})\Lambda g_c{}^{\boldsymbol{b}}(\boldsymbol{k})a_{\boldsymbol{b}}e^{-ik\cdot x}+\text{H.c.} \\
= & \int d\Gamma(\boldsymbol{k})|\boldsymbol{\Lambda^{-1}k}\rangle\langle\boldsymbol{\Lambda^{-1}k}|\otimes\left(k_a(\boldsymbol{k})\Lambda g_b{}^{\boldsymbol{b}}(\boldsymbol{k})-k_b(\boldsymbol{k})\Lambda g_a{}^{\boldsymbol{b}}(\boldsymbol{k})\right)a_{\boldsymbol{b}}e^{-ik\cdot x}+\text{H.c.} \\
= & \int d\Gamma(\boldsymbol{k})\left(\Lambda_a{}^c k_c(\boldsymbol{\Lambda^{-1}k})\Lambda_b{}^d g_d{}^{\boldsymbol{b}}(\boldsymbol{\Lambda^{-1}k})-\Lambda_b{}^d k_d(\boldsymbol{\Lambda^{-1}k})\Lambda_a{}^c g_c{}^{\boldsymbol{b}}(\boldsymbol{\Lambda^{-1}k})\right)a_{\boldsymbol{b}}(\boldsymbol{\Lambda^{-1}k},1)e^{-ik\cdot x}+\text{H.c.} \\
= & \int d\Gamma(\boldsymbol{k})\left(\Lambda_a{}^c k_c(\boldsymbol{k})\Lambda_b{}^d g_d{}^{\boldsymbol{b}}(\boldsymbol{k})-\Lambda_b{}^d k_d(\boldsymbol{k})\Lambda_a{}^c g_c{}^{\boldsymbol{b}}(\boldsymbol{k})\right)a_{\boldsymbol{b}}(\boldsymbol{k},1)e^{-i\Lambda k\cdot x}+\text{H.c.} \\
= & \Lambda_a{}^c\Lambda_b{}^d\int d\Gamma(\boldsymbol{k})\left(k_c(\boldsymbol{k})g_d{}^{\boldsymbol{a}}(\boldsymbol{k})-k_d(\boldsymbol{k})g_c{}^{\boldsymbol{a}}(\boldsymbol{k})\right)a_{\boldsymbol{a}}(\boldsymbol{k},1)e^{-ik\cdot\Lambda^{-1}x}+\text{H.c.} \\
= & \Lambda_a{}^c\Lambda_b{}^d F_{cd}(\Lambda^{-1}x,1).
\end{aligned} \tag{407}
$$

The same can be shown for arbitrary N-oscillator representation.

5.10 Transformation properties of vacuum

Let us remind ourselves that the definition of vacuum in this representation is not unique, i.e.

$$|O(1)\rangle=\int d\Gamma(\boldsymbol{k})O(\boldsymbol{k})|\boldsymbol{k},0,0,0,0\rangle.$$

Then the Lorentz transformation acts on vacuum as follows

$$U(\Lambda,0,1)|O(1)\rangle=U(\Lambda,0,1)\int d\Gamma(\boldsymbol{k})O(\boldsymbol{k})|\boldsymbol{k},0,0,0,0\rangle=\int d\Gamma(\boldsymbol{k})O(\boldsymbol{\Lambda^{-1}k})|\boldsymbol{k},0,0,0,0\rangle. \tag{408}$$

A transformed vacuum state is again a vacuum state, but the probability of finding $\boldsymbol{k}$ is modified by the Doppler effect. The extension to $N>1$ is obvious. As a by product we observe that the vacuum field transforms as a scalar field

$$O(\boldsymbol{k}) \quad\mapsto\quad O(\boldsymbol{\Lambda^{-1}k}). \tag{409}$$

This also implies the following transformation rule of the vacuum probability density

$$Z(\boldsymbol{k}) \quad\mapsto\quad Z(\boldsymbol{\Lambda^{-1}k}). \tag{410}$$

Of course, the norm of such a transformed vacuum is invariant due to the Lorentz invariant measure (28), and therefore

$$\langle O,1|U(\Lambda,0,1)^\dagger U(\Lambda,0,1)|O(1)\rangle \;=\; \int d\Gamma(\boldsymbol{k})|O(\boldsymbol{\Lambda^{-1}k})|^2 \;=\; \int d\Gamma(\boldsymbol{\Lambda k})|O(\boldsymbol{k})|^2 \;=\; 1. \tag{411}$$

5.11 Gauge transformation

Quantum field theory is assumed to be gauge invariant. The change of gauge is a change in electromagnetic potential that does not produce any change in physical observables. In this section it will be shown that for the reducible covariant representation there exists a transformation on the spin-frame level that corresponds to a gauge transformation on the potential level in $\Psi_{EM}(1)$ vector space. $\Psi_{EM}(1)$ space was introduced and discussed earlier in section 4.4.

Now let us start from the potential operator (208) and see how it transforms after a spin-frame transformation:

$$\omega_A(\boldsymbol{k}) \quad \mapsto \quad \tilde{\omega}_A(\boldsymbol{k}) \; = \; \omega_A(\boldsymbol{k}) + \phi(\boldsymbol{k})\pi_A(\boldsymbol{k}).$$

To see how the gauge transformation acts on ladder operators associated with the null tetrad, we can calculate

$$U_G(\boldsymbol{k})^\dagger a_{\boldsymbol{a}'} U_G(\boldsymbol{k}) \;=\; U_G(\boldsymbol{k})^\dagger g_{\boldsymbol{a}'}{}^{\boldsymbol{a}} a_{\boldsymbol{a}} U_G(\boldsymbol{k}) \;=\; g_{\boldsymbol{a}'}{}^{\boldsymbol{a}} G_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\boldsymbol{k}) g_{\boldsymbol{b}}{}^{\boldsymbol{b}'} a_{\boldsymbol{b}'} \;=\; G_{\boldsymbol{a}'}{}^{\boldsymbol{b}'}(\boldsymbol{k}) a_{\boldsymbol{b}'}, \tag{412}$$

so that

$$\begin{aligned}
U_G(\boldsymbol{k})^\dagger a_{00'} U_G(\boldsymbol{k}) &= a_{00'} - \bar{\phi}(\boldsymbol{k}) a_{01'} - \phi(\boldsymbol{k}) a_{10'} + |\phi(\boldsymbol{k})|^2 a_{11'},\\
U_G(\boldsymbol{k})^\dagger a_{01'} U_G(\boldsymbol{k}) &= U_G(\boldsymbol{k})^\dagger a_+ U_G(\boldsymbol{k}) \;=\; a_+ - \phi(\boldsymbol{k}) a_{11'},\\
U_G(\boldsymbol{k})^\dagger a_{10'} U_G(\boldsymbol{k}) &= U_G(\boldsymbol{k})^\dagger a_- U_G(\boldsymbol{k}) \;=\; a_- - \bar{\phi}(\boldsymbol{k}) a_{11'},\\
U_G(\boldsymbol{k})^\dagger a_{11'} U_G(\boldsymbol{k}) &= a_{11'}.
\end{aligned}$$

From formulas (C.2)-(C.9) derived in appendix, we get the transformation rules for the tetrads and this implies the following transformation of the vector potential

$$\begin{aligned}
\tilde{A}_a(x,1) &= i\int d\Gamma(\boldsymbol{k})\left(-\tilde{m}_a(\boldsymbol{k})a_-(\boldsymbol{k},1) - \tilde{\bar{m}}_a(\boldsymbol{k})a_+(\boldsymbol{k},1) - \tilde{z}_a(\boldsymbol{k})a_3(\boldsymbol{k},1) + \tilde{t}_a(\boldsymbol{k})a_0(\boldsymbol{k},1)\right)e^{-ik\cdot x} + \text{H.c.}\\
&= A_a(x,1) + i\int d\Gamma(\boldsymbol{k})\left(-\phi(\boldsymbol{k})k_a(\boldsymbol{k})a_-(\boldsymbol{k},1) - \bar{\phi}(\boldsymbol{k})k_a(\boldsymbol{k})a_+(\boldsymbol{k},1)\right)e^{-ik\cdot x} + \text{H.c.}\\
&- \frac{i}{\sqrt{2}}\int d\Gamma(\boldsymbol{k})\left(\phi(\boldsymbol{k})\bar{m}_a(\boldsymbol{k}) + \bar{\phi}(\boldsymbol{k})m_a(\boldsymbol{k}) + |\phi(\boldsymbol{k})|^2 k_a(\boldsymbol{k})\right)\left(a_3(\boldsymbol{k},1) - a_0(\boldsymbol{k},1)\right)e^{-ik\cdot x} + \text{H.c.}
\end{aligned} \tag{413}$$

Now the new potential operator can be written as

$$\tilde{A}_a(x,1) \;=\; A_a(x,1) + \partial_a\varphi(x,1) + B_a(x,1), \tag{414}$$

where

$$\varphi(x,1) \;=\; \int d\Gamma(\boldsymbol{k})\left(\phi(\boldsymbol{k})a_-(\boldsymbol{k},1) + \bar{\phi}(\boldsymbol{k})a_+(\boldsymbol{k},1)\right)e^{-ik\cdot x} + \text{H.c.}, \tag{415}$$

$$B_a(x,1) \;=\; -\frac{i}{\sqrt{2}}\int d\Gamma(\boldsymbol{k})\left(\phi(\boldsymbol{k})\bar{m}_a(\boldsymbol{k}) + \bar{\phi}(\boldsymbol{k})m_a(\boldsymbol{k}) + |\phi(\boldsymbol{k})|^2 k_a(\boldsymbol{k})\right)\left(a_3(\boldsymbol{k},1) - a_0(\boldsymbol{k},1)\right)e^{-ik\cdot x} + \text{H.c.} \tag{416}$$

So on first sight this transformation is not exactly a gauge transformation because of the $B_a(x,1)$ term, which contains of the "bad ghost" operator. It can be shown though that in $\Psi_{EM}(1)$ vector space the $B_a(x,1)$ contribution vanishes, i.e.

$$\begin{aligned}
\langle\Psi_{EM}(1)|\tilde{A}_a(x,1)|\Psi_{EM}(1)\rangle &= \langle\Psi_{EM}(1)|\left(A_a(x,1) + \partial_a\varphi(x,1) + B_a(x,1)\right)|\Psi_{EM}(1)\rangle\\
&= \langle\Psi_{EM}(1)|\left(A_a(x,1) + \partial_a\varphi(x,1)\right)|\Psi_{EM}(1)\rangle.
\end{aligned} \tag{417}$$

We can also check the Lorenz condition for the potential operator (413) under spin-frame transformation (412)

$$\begin{aligned}
&\partial^a\tilde{A}_a(x,1)\\
&= \partial^a A_a(x,1) + \int d\Gamma(\boldsymbol{k})k^a(\boldsymbol{k})\left(-\phi(\boldsymbol{k})k_a(\boldsymbol{k})a_-(\boldsymbol{k},1) - \bar{\phi}(\boldsymbol{k})k_a(\boldsymbol{k})a_+(\boldsymbol{k},1)\right)e^{-ik\cdot x} + \text{H.c.}\\
&- \frac{i}{\sqrt{2}}\int d\Gamma(\boldsymbol{k})k^a(\boldsymbol{k})\left(\phi(\boldsymbol{k})\bar{m}_a(\boldsymbol{k}) + \bar{\phi}(\boldsymbol{k})m_a(\boldsymbol{k}) + |\phi(\boldsymbol{k})|^2 k_a(\boldsymbol{k})\right)\left(a_3(\boldsymbol{k},1) - a_0(\boldsymbol{k},1)\right)e^{-ik\cdot x} + \text{H.c.}\\
&= \partial^a A_a(x,1).
\end{aligned} \tag{418}$$

As one can see, if the four-vector potential (208) holds the Lorenz condition, so does (413). Earlier, in section 4.4, it was shown that there exists a space denoted by $\Psi_{EM}(1)$ in which a weaker Lorenz condition (217) holds. Therefore, transformation (412) on the spin-frame level corresponds to the Lorenz gauge transformation on the potential level in $\Psi_{EM}(1)$ vector space.

This result should be compared with those of Janner and Janssen form 1971 [21] followed by Han, Kim and Son paper [23]. Although they did not work with a covariant potential operator with four polarization degrees of freedom, they worked out a similar conclusion for the potential operator (A_0, A_1, A_2, A_3), where $A_0 = A_3$. The conclusion was that L_1 and L_2 generators carry gauge transformations of the potential. Here the conclusion is the same for equal numbers of longitudinal and time-like photons, i.e. with $n_0 = n_3$.

For the electromagnetic field operator after spin-frame transformation (412) we get:

$$\begin{aligned}\tilde{F}_{ab}(x,1) &= \int d\Gamma(\boldsymbol{k})\pi_A(\boldsymbol{k})\pi_B(\boldsymbol{k})\varepsilon_{A'B'}\left(a_-(\boldsymbol{k},1)e^{-ik\cdot x} + a_+(\boldsymbol{k},1)^\dagger e^{ik\cdot x}\right) + \text{H.c.}\\ &+ \frac{1}{\sqrt{2}}\int d\Gamma(\boldsymbol{k})^*\tilde{M}_{ab}(\boldsymbol{k})\left(a_3(\boldsymbol{k},1) - a_0(\boldsymbol{k},1)\right)e^{-ik\cdot x} + \text{H.c.}\end{aligned} \tag{419}$$

Here

$$\begin{aligned}{}^*\tilde{M}_{ab}(\boldsymbol{k}) &= \tilde{\omega}_a(\boldsymbol{k})k_b(\boldsymbol{k}) - \tilde{\omega}_b(\boldsymbol{k})k_a(\boldsymbol{k})\\ &= {}^*M_{ab}(\boldsymbol{k}) + \bar{\phi}(\boldsymbol{k})\left(m_a(\boldsymbol{k})k_b(\boldsymbol{k}) - m_b(\boldsymbol{k})k_a(\boldsymbol{k})\right) + \phi(\boldsymbol{k})\left(\bar{m}_a(\boldsymbol{k})k_b(\boldsymbol{k}) - \bar{m}_b(\boldsymbol{k})k_a(\boldsymbol{k})\right).\end{aligned} \tag{420}$$

This would mean that the electromagnetic field in not invariant under this gauge symmetry. Again it can be shown that in $\Psi_{EM}(1)$ average the ${}^*\tilde{M}_{ab}(\boldsymbol{k})$ contribution vanishes leaving the electromagnetic field unchanged.

$$\langle\Psi_{EM}(1)|\tilde{F}_{ab}(x,1)|\Psi_{EM}(1)\rangle = \langle\Psi_{EM}(1)|F_{ab}(x,1)|\Psi_{EM}(1)\rangle. \tag{421}$$

5.12 Invariants in a combined homogeneous Lorentz and gauge transformation

As one can see in (354-357) transformation (359) mixes annihilation operators $a_1,\ a_2$ with excitations $a_3,\ a_0$. Reminding:

$$\begin{aligned}U(\Theta,\phi)^\dagger a_0 U(\Theta,\phi) &= -|\phi|\cos(\xi+2\Theta)a_1 + |\phi|\sin(\xi+2\Theta)a_2 - \frac{|\phi|^2}{2}a_3 + \left(1+\frac{|\phi|^2}{2}\right)a_0,\\ U(\Theta,\phi)^\dagger a_1 U(\Theta,\phi) &= \cos 2\Theta a_1 - \sin 2\Theta a_2 + |\phi|\cos\xi a_3 - |\phi|\cos\xi a_0,\\ U(\Theta,\phi)^\dagger a_2 U(\Theta,\phi) &= \sin 2\Theta a_1 + \cos 2\Theta a_2 - |\phi|\sin\xi a_3 + |\phi|\sin\xi a_0,\\ U(\Theta,\phi)^\dagger a_3 U(\Theta,\phi) &= -|\phi|\cos(\xi+2\Theta)a_1 + |\phi|\sin(\xi+2\Theta)a_2 + \left(1-\frac{|\phi|^2}{2}\right)a_3 + \frac{|\phi|^2}{2}a_0.\end{aligned}$$

If we considered $|\phi| = 0$ we would get a pure homogeneous Lorentz transformation on operators a_1 and a_2 corresponding to transverse polarization degrees of freedom. Let us also take a closer look at the "bad ghost" operator $a_{11'} = (a_3 - a_0)/\sqrt{2}$ and how it transforms under transformation (359). For this purpose we will consider ladder operators corresponding to the null tetrad in the potential operator (211). Then the transformation is

$$U(\Theta,\phi)^\dagger a_{00'} U(\Theta,\phi) = a_{00'} - |\phi(\boldsymbol{k})|e^{-i(\xi(\boldsymbol{k})+2\Theta(\Lambda,\boldsymbol{k}))}a_{01'} - |\phi(\boldsymbol{k})|e^{i(\xi(\boldsymbol{k})+2\Theta(\Lambda,\boldsymbol{k}))}a_{10'} + |\phi(\boldsymbol{k})|^2 a_{11'}, \tag{422}$$

$$U(\Theta,\phi)^\dagger a_{01'} U(\Theta,\phi) = e^{-2i\Theta(\Lambda,\boldsymbol{k})}a_{01'} - |\phi(\boldsymbol{k})|e^{i\xi(\boldsymbol{k})}a_{11'}, \tag{423}$$

$$U(\Theta,\phi)^\dagger a_{10'} U(\Theta,\phi) = e^{2i\Theta(\Lambda,\boldsymbol{k})}a_{10'} - |\phi(\boldsymbol{k})|e^{-i\xi(\boldsymbol{k})}a_{11'}, \tag{424}$$

$$U(\Theta,\phi)^\dagger a_{11'} U(\Theta,\phi) = a_{11'}. \tag{425}$$

As one can see transformation (359) does not change the "bad ghost" operator $a_3 - a_0$, i.e.

$$U(\Theta,\phi)^\dagger (a_3 - a_0) U(\Theta,\phi) = a_3 - a_0. \tag{426}$$

It is easy to show that the covariant total number of photons does not change due to the combined Lorentz and gauge transformation, i.e.

$$\begin{aligned}
& U(\Theta,\phi)^\dagger\,(n_1+n_2+n_3-n_0)\,U(\Theta,\phi) \;=\; U(\Theta,\phi)^\dagger\left(a_1^\dagger a_1+a_2^\dagger a_2+a_3^\dagger a_3-a_0 a_0^\dagger\right)U(\Theta,\phi)\\
=\;& U(\Theta,\phi)^\dagger\left(a_1^\dagger a_1+a_2^\dagger a_2+a_3^\dagger a_3-a_0^\dagger a_0+1\right)U(\Theta,\phi) \;=\; U(\Theta,\phi)^\dagger\left(-a^{\boldsymbol{a}\dagger}a_{\boldsymbol{a}}+1\right)U(\Theta,\phi)\\
=\;& -L(\Lambda,\boldsymbol{k})^{\boldsymbol{a}}{}_{\boldsymbol{b}}L(\Lambda,\boldsymbol{k})_{\boldsymbol{a}}{}^{\boldsymbol{c}}a^{\boldsymbol{b}\dagger}a_{\boldsymbol{c}}+1 \;=\; -\,g_{\boldsymbol{b}}{}^{\boldsymbol{c}}a^{\boldsymbol{b}\dagger}a_{\boldsymbol{c}}+1 \;=\; -\,a^{\boldsymbol{b}\dagger}a_{\boldsymbol{b}}+1. \qquad (427)
\end{aligned}$$

5.13 Four-translations in four-dimensional oscillator representation

Let us begin with the $N=1$ oscillator representation and denote $U(\mathbf{1},y,1)=e^{iP(1)\cdot y}$. The generator of four-translations, the four-momentum reads

$$\begin{aligned}
P_{\boldsymbol{a}}(1) &= \int d\Gamma(\boldsymbol{k})k_{\boldsymbol{a}}|\boldsymbol{k}\rangle\langle\boldsymbol{k}|\otimes\left(a_1^\dagger a_1+a_2^\dagger a_2+a_3^\dagger a_3-a_0^\dagger a_0+2\right)\\
&= \int d\Gamma(\boldsymbol{k})k_{\boldsymbol{a}}\big(n_1(\boldsymbol{k},1)+n_2(\boldsymbol{k},1)+n_3(\boldsymbol{k},1)-n_0(\boldsymbol{k},1)+1\big)\\
&= \int d\Gamma(\boldsymbol{k})k_{\boldsymbol{a}}\big(n_+(\boldsymbol{k},1)+n_-(\boldsymbol{k},1)+n_3(\boldsymbol{k},1)-n_0(\boldsymbol{k},1)+1\big), \qquad (428)
\end{aligned}$$

where $P_0(1)$ is of course the Hamiltonian (183) introduced earlier in chapter 4. One can immediately verify that

$$e^{iP(1)\cdot y}a_1(\boldsymbol{k},1)e^{-iP(1)\cdot y} = a_1(\boldsymbol{k},1)e^{-iy\cdot k}, \qquad (429)$$
$$e^{iP(1)\cdot y}a_2(\boldsymbol{k},1)e^{-iP(1)\cdot y} = a_2(\boldsymbol{k},1)e^{-iy\cdot k}, \qquad (430)$$
$$e^{iP(1)\cdot y}a_3(\boldsymbol{k},1)e^{-iP(1)\cdot y} = a_3(\boldsymbol{k},1)e^{-iy\cdot k}, \qquad (431)$$
$$e^{iP(1)\cdot y}a_0(\boldsymbol{k},1)e^{-iP(1)\cdot y} = a_0(\boldsymbol{k},1)e^{-iy\cdot k}, \qquad (432)$$

implying the following transformation on the vector potential

$$U(\mathbf{1},y,1)^\dagger A_a(x,1)U(\mathbf{1},y,1) = A_a(x-y,1). \qquad (433)$$

Furthermore, the four-momentum for arbitrary N-oscillator reads

$$\begin{aligned}
P_{\boldsymbol{a}}(N) &= \sum_{n=1}^{N}P_{\boldsymbol{a}}(1)^{(n)}\\
&= \int d\Gamma(\boldsymbol{k})k_{\boldsymbol{a}}\big(n_1(\boldsymbol{k},N)+n_2(\boldsymbol{k},N)+n_3(\boldsymbol{k},N)-n_0(\boldsymbol{k},N)+1\big)\\
&= \int d\Gamma(\boldsymbol{k})k_{\boldsymbol{a}}\big(n_+(\boldsymbol{k},N)+n_-(\boldsymbol{k},N)+n_3(\boldsymbol{k},N)-n_0(\boldsymbol{k},N)+1\big). \qquad (434)
\end{aligned}$$

Vectors (193) are simultaneously the eigenvectors of $P_{\boldsymbol{a}}(N)$, i.e.

$$\begin{aligned}
& P^{\boldsymbol{a}}(N)|\boldsymbol{k}_1,\dots,\boldsymbol{k}_N,n_0^1,\dots,n_3^N\rangle\\
=\;& \Big(k_1^{\boldsymbol{a}}\big(n_1^1+n_2^1+n_3^1-n_0^1+1\big)+\dots+k_N^{\boldsymbol{a}}\big(n_1^N+n_2^N+n_3^N-n_0^N+1\big)\Big)\\
\times\;& |\boldsymbol{k}_1,\dots,\boldsymbol{k}_N,n_0^1,\dots,n_3^N\rangle. \qquad (435)
\end{aligned}$$

Then the following transformation rule for the ladder operators in the N-oscillator reducible representation holds

$$e^{iP(N)\cdot y}a_1(\boldsymbol{k},N)e^{-iP(N)\cdot y} = a_1(\boldsymbol{k},N)e^{-iy\cdot k}, \qquad (436)$$
$$e^{iP(N)\cdot y}a_2(\boldsymbol{k},N)e^{-iP(N)\cdot y} = a_2(\boldsymbol{k},N)e^{-iy\cdot k}, \qquad (437)$$
$$e^{iP(N)\cdot y}a_3(\boldsymbol{k},N)e^{-iP(N)\cdot y} = a_3(\boldsymbol{k},N)e^{-iy\cdot k}, \qquad (438)$$
$$e^{iP(N)\cdot y}a_0(\boldsymbol{k},N)e^{-iP(N)\cdot y} = a_0(\boldsymbol{k},N)e^{-iy\cdot k}, \qquad (439)$$

implying

$$U(\mathbf{1},y,N)^\dagger A_a(x,N)U(\mathbf{1},y,N) = A_a(x-y,N). \qquad (440)$$

The Poincaré group, i.e. the semi-direct product of homogeneous Lorentz transformation and space-time translation groups is

$$U(\Lambda, y, 1) \;=\; U(\mathbf{1}, y, 1)U(\Lambda, 0, 1), \tag{441}$$

and the composition law of two successive Poincaré transformations holds

$$U(\Lambda_2, y_2, 1)U(\Lambda_1, y_1, 1) \;=\; U(\Lambda_2\Lambda_1, \Lambda_2 y_1 + y_2, 1). \tag{442}$$

5.14 Conclusions and results

Most of the results presented in this chapter were already published in [12] and [13]. The notation, starting from section 5.1, is set differently here, in a way that the homogeneous Lorentz and gauge transformations are treated as separate non-commuting transformations. Further, generators of these transformations coming from the canonical variables, introduced earlier in chapter 4, are shown. In next sections the composition law for homogeneous Lorentz transformation and the additivity of Lorentz transformation on the gauge parameter are proved. These are new results. In 5.8, 5.9 and 5.10 the homogeneous Lorentz transformation acting on the four-vector potential, electromagnetic field operator and vacuum are shown respectively. As a by product of such a transformation acting on a non-unique vacuum we observe that the vacuum field transforms as a scalar field. In 5.11 it has been pointed out that for the reducible covariant representation there exists a transformation on the spin-frame level that corresponds to a gauge transformation on the potential level for Ψ_{EM} vectors introduced earlier. In section 5.12 invariants of introduced transformations are shown. Let us stress that the "ghost operator" coming from the two extra degrees of photon polarization is an invariant. Finally in 5.13 the four-translations for the four-dimensional oscillator representation are introduced.

6 Two-photon field

The purpose for this chapter is to develop a model for the four Bell states in the reducible representation of N-oscillator algebras. First in section 6.1 a notation for a two-photon field operator will be shown. Further in section 6.2 a model for the four Bell state photon field operators is developed. The main assumption made here, following Zeilinger's paper [44], is that Bell states are states maximally correlated in both bases: linear and circular.

6.1 Two-photon field operator

Let us first note that the correlation of two-photon states in N-oscillator reducible representation does not come straightforward form the tensor product. To see this, let us first rewrite the ladder operators in N-oscillator representation, i.e.

$$\begin{aligned}
a_s(\boldsymbol{k},N)^\dagger a_{s'}(\boldsymbol{k}',N)^\dagger &= \frac{1}{\sqrt{N}}\sum_n^N a_s(\boldsymbol{k},1)^{\dagger(n)}\frac{1}{\sqrt{N}}\sum_m^N a_{s'}(\boldsymbol{k}',1)^{\dagger(m)}\\
&= \frac{1}{N}\sum_{n=m}^N a_s(\boldsymbol{k},1)^{\dagger(n)}a_{s'}(\boldsymbol{k}',1)^{\dagger(m)} + \frac{1}{N}\sum_{n\neq m}^N a_s(\boldsymbol{k},1)^{\dagger(n)}a_{s'}(\boldsymbol{k}',1)^{\dagger(m)}\\
&= \frac{1}{N}\underbrace{\left(a_s(\boldsymbol{k},1)^\dagger a_{s'}(\boldsymbol{k}',1)^\dagger\otimes ...\otimes I(1)+...+I(1)\otimes...\otimes a_s(\boldsymbol{k},1)^\dagger a_{s'}(\boldsymbol{k}',1)^\dagger\right)}_{N}\\
&+ \frac{1}{N}\underbrace{\left(a_s(\boldsymbol{k},1)^\dagger\otimes a_{s'}(\boldsymbol{k}',1)^\dagger\otimes ...\otimes I(1)+...+I(1)\otimes...\otimes a_s(\boldsymbol{k},1)^\dagger \otimes a_{s'}(\boldsymbol{k}',1)^\dagger\right)}_{N^2-N}.
\end{aligned} \tag{443}$$

As one can see, in (443) we have N terms where the creation operators live on the same oscillator and N^2-N terms where they live on separate oscillators. Concerning tensor algebra representations, it is worth mentioning that the choice the algebra representation may become important also in other contexts. For example, it was shown by Pawłowski and Czachor [15] that "entanglement with vacuum" turns out to be a notion that depends on representation.

Now let us consider a whole frequency spectrum two-photon field operator in circular basis for the N-oscillator reducible representation

$$\Psi(N) = \sum_{s,s'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \psi_{ss'}(\boldsymbol{k},\boldsymbol{k}')a_s(\boldsymbol{k},N)^\dagger a_{s'}(\boldsymbol{k}',N)^\dagger. \tag{444}$$

If this operator can be factored into two operators such that

$$\Psi(N) = \sum_{s=\pm}\int d\Gamma(\boldsymbol{k})\ g_s(\boldsymbol{k})a_s(\boldsymbol{k},N)^\dagger \times \sum_{s'=\pm}\int d\Gamma(\boldsymbol{k}')\ h_{s'}(\boldsymbol{k}')a_{s'}(\boldsymbol{k}',N)^\dagger, \tag{445}$$

we will call it a separable field operator. To study in more detail the symmetry properties of such two-photon operators we write:

$$\Psi(N) = \sum_{s,s'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \frac{\psi_{ss'}(\boldsymbol{k},\boldsymbol{k}')+\psi_{s's}(\boldsymbol{k}',\boldsymbol{k})}{2}a_s(\boldsymbol{k},N)^\dagger a_{s'}(\boldsymbol{k}',N)^\dagger. \tag{446}$$

Let us also note that, from the integral's (446) symmetry properties, it follows

$$\psi_{ss'}(\boldsymbol{k},\boldsymbol{k}') = \psi_{s's}(\boldsymbol{k}',\boldsymbol{k}). \tag{447}$$

From the relation between circular and linear polarizations derived in the appendix (F.40), we can get to the linear polarization basis:

$$\begin{aligned}
\Psi(N) &= \sum_{s,s'}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{i(s\theta(\boldsymbol{k})+s'\theta(\boldsymbol{k}'))}\frac{\psi_{ss'}(\boldsymbol{k},\boldsymbol{k}')}{2}\left(a_\theta(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger - ss'a_{\theta'}(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger\right) \\
&+ \sum_{s,s'}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{i(s\theta(\boldsymbol{k})+s'\theta(\boldsymbol{k}'))}\frac{\psi_{ss'}(\boldsymbol{k},\boldsymbol{k}')}{2}\left(is'a_\theta(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger + isa_{\theta'}(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger\right) \\
&\qquad (448) \\
&= \sum_{s,s'}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{i(s\theta(\boldsymbol{k})+s'\theta(\boldsymbol{k}'))}\frac{\psi_{ss'}(\boldsymbol{k},\boldsymbol{k}')}{2}\left(a_\theta(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger - ss'a_{\theta'}(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger\right) \\
&+ i\sum_{s,s'}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ s'e^{i(s\theta(\boldsymbol{k})+s'\theta(\boldsymbol{k}'))}\psi_{ss'}(\boldsymbol{k},\boldsymbol{k}')a_\theta(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger. \qquad (449)
\end{aligned}$$

Then the scalar product of such two-photon state reads

$$\begin{aligned}
&\langle O(N)|\Psi(N)^\dagger\Psi(N)|O(N)\rangle \\
&= \frac{2}{N}\sum_{s,s'=\pm}\int d\Gamma(\boldsymbol{k})|\psi_{ss'}(\boldsymbol{k},\boldsymbol{k})|^2 Z(\boldsymbol{k}) + \frac{2(N-1)}{N}\sum_{s,s'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|\psi_{ss'}(\boldsymbol{k},\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}'). \qquad (450)
\end{aligned}$$

This formula is derived step by step in appendix (J.2). Also in the $N\to\infty$ limit we get

$$\lim_{N\to\infty}\langle O(N)|\Psi(N)^\dagger\Psi(N)|O(N)\rangle \ = \ 2\sum_{s,s'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|\psi_{ss'}(\boldsymbol{k},\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}'). \qquad (451)$$

As one can see, the first term of the scalar product (450), i.e.

$$\sum_{s,s'=\pm}\int d\Gamma(\boldsymbol{k})|\psi_{ss'}(\boldsymbol{k},\boldsymbol{k})|^2 Z(\boldsymbol{k}) \qquad (452)$$

does not occur in $N\to\infty$ limit.

6.2 Bell state field operators

It was in 1964 when John Bell proved his theorem allowing the experimental test of whether Einstein's spooky actions at-a-distance exist. In quantum mechanics particles are called entangled, if their state can not be factored into single particle states. Experimentally one of such states can be generated by a parametric down conversion in a nonlinear crystal and the other three Bell states can be obtained by suitable unitary operators with linear polarization elements.

Now let us study some cases of maximal photon correlations in a quantum field theory background for the reducible N-oscillator representation. First we will consider photons in circular basis that are anti-correlated: one is left-handed the other right-handed. Anti-correlated in circular basis field operators will be here denoted by $\Psi_1(N)$, so that

$$\begin{aligned}
\Psi_1(N) &= \sum_{s\neq s'}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \psi_{ss'}(\boldsymbol{k},\boldsymbol{k}')a_s(\boldsymbol{k},N)^\dagger a_{s'}(\boldsymbol{k}',N)^\dagger \\
&= \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \left(\psi_{+-}(\boldsymbol{k},\boldsymbol{k}')a_+(\boldsymbol{k},N)^\dagger a_-(\boldsymbol{k}',N)^\dagger + \psi_{-+}(\boldsymbol{k},\boldsymbol{k}')a_-(\boldsymbol{k},N)^\dagger a_+(\boldsymbol{k}',N)^\dagger\right) \qquad (453) \\
&= \sum_{s\neq s'}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{i(s\theta(\boldsymbol{k})+s'\theta(\boldsymbol{k}'))}\frac{\psi_{ss'}(\boldsymbol{k},\boldsymbol{k}')}{2}\left(a_\theta(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger + a_{\theta'}(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger\right) \\
&+ i\sum_{s\neq s'}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ s'e^{i(s\theta(\boldsymbol{k})+s'\theta(\boldsymbol{k}'))}\psi_{ss'}(\boldsymbol{k},\boldsymbol{k}')a_\theta(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger. \qquad (454)
\end{aligned}$$

This field operator can not be factored like (445). We can say about $\Psi_1(N)$: both photons have different polarizations in circular basis. For the maximal anti-correlation the condition on the field must hold

$|\psi_{+-}(\boldsymbol{k},\boldsymbol{k}')|^2 = |\psi_{-+}(\boldsymbol{k},\boldsymbol{k}')|^2$. To fully describe four Bell states we need a second basis. This is why we write the $\Psi_1(N)$ field operator in two bases: circular (453) and linear (454). Let us stress that for the linear basis the polarization angle $\theta(\boldsymbol{k})$ is assumed to be dependent on momentum. There are two situations when such states are still maximally correlated in the second, here linear basis. The first case is when photons in linear basis are maximally anti-correlated. Let us denote by $\theta_{11}(\boldsymbol{k})$ the polarization function for such states. Then, from (454), the following condition on the field and polarization angle must hold:

$$\sum_{s=\pm} e^{is(\theta_{11}(\boldsymbol{k})-\theta_{11}(\boldsymbol{k}'))}\psi_{s-s}(\boldsymbol{k},\boldsymbol{k}') = 0 \quad \Rightarrow \quad e^{i(\theta_{11}(\boldsymbol{k})-\theta_{11}(\boldsymbol{k}'))}\psi_{+-}(\boldsymbol{k},\boldsymbol{k}') = -e^{i(\theta_{11}(\boldsymbol{k}')-\theta_{11}(\boldsymbol{k}))}\psi_{-+}(\boldsymbol{k},\boldsymbol{k}'), \tag{455}$$

and such a field operator, denoted here by $\Psi_{11}(N)$, may be written in forms

$$\begin{aligned}
\Psi_{11}(N) &= \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \left(\psi_{+-}(\boldsymbol{k},\boldsymbol{k}')a_+(\boldsymbol{k},N)^\dagger a_-(\boldsymbol{k}',N)^\dagger + \psi_{-+}(\boldsymbol{k},\boldsymbol{k}')a_-(\boldsymbol{k},N)^\dagger a_+(\boldsymbol{k}',N)^\dagger\right)\\
&= -i\sum_{s=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ se^{i(s\theta_{11}(\boldsymbol{k})-s\theta_{11}(\boldsymbol{k}'))}\psi_{s-s}(\boldsymbol{k},\boldsymbol{k}')a_\theta(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger\\
&= -2i\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{i(\theta_{11}(\boldsymbol{k})-\theta_{11}(\boldsymbol{k}'))}\psi_{+-}(\boldsymbol{k},\boldsymbol{k}')a_\theta(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger\\
&= 2i\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{-i(\theta_{11}(\boldsymbol{k})-\theta_{11}(\boldsymbol{k}'))}\psi_{-+}(\boldsymbol{k},\boldsymbol{k}')a_\theta(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger.
\end{aligned} \tag{456}$$

From (455) we see that if we want to have a two-photon state maximally correlated in both bases, an implicit relation must take place that relates the fields $\psi_{+-}(\boldsymbol{k},\boldsymbol{k}')$ with the polarization angles. Now we can say about $\Psi_{11}(N)$: both photons have different polarizations in circular and linear basis. In such a case operator (456) will represent one of the four Bell states. Then the inner product reads

$$\begin{aligned}
&\langle O(N)|\Psi_{11}(N)^\dagger\Psi_{11}(N)|O(N)\rangle\\
=\ &\frac{4}{N}\int d\Gamma(\boldsymbol{k})|\psi_{+-}(\boldsymbol{k},\boldsymbol{k})|^2 Z(\boldsymbol{k}) + \frac{4(N-1)}{N}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|\psi_{+-}(\boldsymbol{k},\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}'),
\end{aligned} \tag{457}$$

and in the $N\to\infty$ limit we have

$$\lim_{N\to\infty}\langle O(N)|\Psi_{11}(N)^\dagger\Psi_{11}(N)|O(N)\rangle = 4\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|\psi_{+-}(\boldsymbol{k},\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}'). \tag{458}$$

The second case is when photons are maximally correlated in linear basis. Then from (454) a condition on the fields and the polarization angles must hold:

$$\begin{aligned}
&\sum_{s=\pm} se^{i(s\theta_{12}(\boldsymbol{k})-s\theta_{12}(\boldsymbol{k}'))}\psi_{s-s}(\boldsymbol{k},\boldsymbol{k}') = 0\\
\Rightarrow\quad &e^{i(\theta_{12}(\boldsymbol{k})-\theta_{12}(\boldsymbol{k}'))}\psi_{+-}(\boldsymbol{k},\boldsymbol{k}') = e^{-i(\theta_{12}(\boldsymbol{k})-\theta_{12}(\boldsymbol{k}'))}\psi_{-+}(\boldsymbol{k},\boldsymbol{k}').
\end{aligned} \tag{459}$$

Here we denote $\theta_{12}(\boldsymbol{k})$ as the polarization function for such a field operator and $\Psi_{12}(N)$ as the field operator corresponding to the Bell state that is anti-correlated in circular basis and correlated in linear one, i.e.

$$\begin{aligned}
\Psi_{12}(N) &= \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \left(\psi_{+-}(\boldsymbol{k},\boldsymbol{k}')a_+(\boldsymbol{k},N)^\dagger a_-(\boldsymbol{k}',N)^\dagger + \psi_{-+}(\boldsymbol{k},\boldsymbol{k}')a_-(\boldsymbol{k},N)^\dagger a_+(\boldsymbol{k}',N)^\dagger\right)\\
&= \sum_{s=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{i(s\theta_{12}(\boldsymbol{k})-s\theta_{12}(\boldsymbol{k}'))}\frac{\psi_{s-s}(\boldsymbol{k},\boldsymbol{k}')}{2}\left(a_\theta(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger + a_{\theta'}(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger\right)\\
&= \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{i(\theta_{12}(\boldsymbol{k})-\theta_{12}(\boldsymbol{k}'))}\psi_{+-}(\boldsymbol{k},\boldsymbol{k}')\left(a_\theta(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger + a_{\theta'}(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger\right)\\
&= \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{-i(\theta_{12}(\boldsymbol{k})-\theta_{12}(\boldsymbol{k}'))}\psi_{-+}(\boldsymbol{k},\boldsymbol{k}')\left(a_\theta(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger + a_{\theta'}(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger\right).
\end{aligned} \tag{460}$$

Also from conditions (455) and (459) we get the following relation for the polarization angles

$$\theta_{11}(\boldsymbol{k}) - \theta_{11}(\boldsymbol{k}') = \theta_{12}(\boldsymbol{k}) - \theta_{12}(\boldsymbol{k}') + \frac{\pi}{2} + n\pi, \qquad n\in\boldsymbol{Z}. \tag{461}$$

This formula seems to be more intuitive than conditions (455) and (459). It simply shows that the difference of the polarization angles for anti-correlated in linear polarizations states is equal to the difference of the polarization angles for correlated states plus a $\pi/2$ factor. Now we can say about $\Psi_{12}(N)$: both photons have different polarizations in circular basis and the same polarizations in linear basis. The inner product for (460) state reads

$$\begin{aligned}
& \langle O(N)|\Psi_{12}(N)^\dagger \Psi_{12}(N)|O(N)\rangle \\
= \; & \frac{4}{N}\int d\Gamma(\boldsymbol{k})|\psi_{+-}(\boldsymbol{k},\boldsymbol{k})|^2 Z(\boldsymbol{k}) + \frac{4(N-1)}{N}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|\psi_{+-}(\boldsymbol{k},\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}'),
\end{aligned} \tag{462}$$

and in the $N\to\infty$ limit we have

$$\lim_{N\to\infty}\langle O(N)|\Psi_{12}(N)^\dagger \Psi_{12}(N)|O(N)\rangle = 4\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|\psi_{+-}(\boldsymbol{k},\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}'). \tag{463}$$

Now let us consider photons in circular basis that are maximally correlated, i.e. $|\psi_{++}(\boldsymbol{k},\boldsymbol{k}')|^2 = |\psi_{--}(\boldsymbol{k},\boldsymbol{k}')|^2$; they are both either left- or right-handed. We will denote such field operators as $\Psi_2(N)$ and write them in both bases

$$\begin{aligned}
\Psi_2(N) &= \sum_{s=s'}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\;\psi_{ss'}(\boldsymbol{k},\boldsymbol{k}')a_s(\boldsymbol{k},N)^\dagger a_{s'}(\boldsymbol{k}',N)^\dagger \\
&= \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\;\left(\psi_{++}(\boldsymbol{k},\boldsymbol{k}')a_+(\boldsymbol{k},N)^\dagger a_+(\boldsymbol{k}',N)^\dagger + \psi_{--}(\boldsymbol{k},\boldsymbol{k}')a_-(\boldsymbol{k},N)^\dagger a_-(\boldsymbol{k}',N)^\dagger\right) \qquad (464) \\
&= \sum_{s=s'}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\;e^{i(s\theta(\boldsymbol{k})+s'\theta(\boldsymbol{k}'))}\frac{\psi_{ss'}(\boldsymbol{k},\boldsymbol{k}')}{2}\left(a_\theta(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger - a_{\theta'}(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger\right) \\
&+ i\sum_{s=s'}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\;s' e^{i(s\theta(\boldsymbol{k})+s'\theta(\boldsymbol{k}'))}\psi_{ss'}(\boldsymbol{k},\boldsymbol{k}')a_\theta(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger. \qquad (465)
\end{aligned}$$

Again there are two situations when such a field operator is still maximally correlated in another, here linear basis. When such photons in linear basis are maximally anti-correlated, a condition on fields $\psi_{--}(\boldsymbol{k},\boldsymbol{k}')$ and $\psi_{++}(\boldsymbol{k},\boldsymbol{k}')$ and the polarization angle denoted here by $\theta_{21}(\boldsymbol{k})$ must hold

$$\begin{aligned}
\sum_{s=\pm} e^{i(s\theta_{21}(\boldsymbol{k})+s\theta_{21}(\boldsymbol{k}'))}\psi_{ss}(\boldsymbol{k},\boldsymbol{k}') &= 0 \\
\Rightarrow \quad e^{i(\theta_{21}(\boldsymbol{k})+\theta_{21}(\boldsymbol{k}'))}\psi_{++}(\boldsymbol{k},\boldsymbol{k}') &= -e^{-i(\theta_{21}(\boldsymbol{k})+\theta_{21}(\boldsymbol{k}'))}\psi_{--}(\boldsymbol{k},\boldsymbol{k}').
\end{aligned} \tag{466}$$

Then the field operator $\Psi_{21}(N)$ can be written in the following forms

$$\begin{aligned}
\Psi_{21}(N) &= \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\;\left(\psi_{++}(\boldsymbol{k},\boldsymbol{k}')a_+(\boldsymbol{k},N)^\dagger a_+(\boldsymbol{k}',N)^\dagger + \psi_{--}(\boldsymbol{k},\boldsymbol{k}')a_-(\boldsymbol{k},N)^\dagger a_-(\boldsymbol{k}',N)^\dagger\right) \\
&= i\sum_{s=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\;s e^{i(s\theta_{21}(\boldsymbol{k})+s\theta_{21}(\boldsymbol{k}'))}\psi_{ss}(\boldsymbol{k},\boldsymbol{k}')a_\theta(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger \\
&= 2i\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\;e^{i(\theta_{21}(\boldsymbol{k})+\theta_{21}(\boldsymbol{k}'))}\psi_{++}(\boldsymbol{k},\boldsymbol{k}')a_\theta(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger \\
&= -2i\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\;e^{-i(\theta_{21}(\boldsymbol{k})+\theta_{21}(\boldsymbol{k}'))}\psi_{--}(\boldsymbol{k},\boldsymbol{k}')a_\theta(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger.
\end{aligned} \tag{467}$$

We can say about $\Psi_{21}(N)$: both photons have the same polarizations in circular and different polarizations in linear basis. Then the inner product reads

$$\begin{aligned}
& \langle O(N)|\Psi_{21}(N)^\dagger \Psi_{21}(N)|O(N)\rangle \\
= \; & \frac{4}{N}\int d\Gamma(\boldsymbol{k})|\psi_{++}(\boldsymbol{k},\boldsymbol{k})|^2 Z(\boldsymbol{k}) + \frac{4(N-1)}{N}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|\psi_{++}(\boldsymbol{k},\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}'),
\end{aligned} \tag{468}$$

and for the $N\to\infty$ limit we have

$$\lim_{N\to\infty}\langle O(N)|\Psi_{21}(N)^\dagger \Psi_{21}(N)|O(N)\rangle = 4\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|\psi_{++}(\boldsymbol{k},\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}'). \tag{469}$$

Finally when photons in linear basis are maximally correlated, a condition on the fields and polarization angles, denoted here by $\theta_{22}(\boldsymbol{k})$, must hold

$$\begin{aligned}\sum_{s=\pm} s e^{i(s\theta_{22}(\boldsymbol{k})+s\theta_{22}(\boldsymbol{k}'))}\psi_{ss}(\boldsymbol{k},\boldsymbol{k}') &= 0 \\ \Rightarrow \quad e^{i(\theta_{22}(\boldsymbol{k})+\theta_{22}(\boldsymbol{k}'))}\psi_{++}(\boldsymbol{k},\boldsymbol{k}') &= e^{-i(\theta_{22}(\boldsymbol{k})+\theta_{22}(\boldsymbol{k}'))}\psi_{--}(\boldsymbol{k},\boldsymbol{k}'),\end{aligned} \tag{470}$$

and the field operator denoted here by $\Psi_{22}(N)$ is

$$\begin{aligned}\Psi_{22}(N) &= \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \left(\psi_{++}(\boldsymbol{k},\boldsymbol{k}')a_+(\boldsymbol{k},N)^\dagger a_+(\boldsymbol{k}',N)^\dagger + \psi_{--}(\boldsymbol{k},\boldsymbol{k}')a_-(\boldsymbol{k},N)^\dagger a_-(\boldsymbol{k}',N)^\dagger\right) \\ &= \sum_{s=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{i(s\theta_{22}(\boldsymbol{k})+s\theta_{22}(\boldsymbol{k}'))}\frac{\psi_{ss}(\boldsymbol{k},\boldsymbol{k}')}{2}\left(a_\theta(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger - a_{\theta'}(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger\right) \\ &= \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{i(\theta_{22}(\boldsymbol{k})+\theta_{22}(\boldsymbol{k}'))}\psi_{++}(\boldsymbol{k},\boldsymbol{k}')\left(a_\theta(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger - a_{\theta'}(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger\right) \\ &= \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{-i(\theta_{22}(\boldsymbol{k})+\theta_{22}(\boldsymbol{k}'))}\psi_{--}(\boldsymbol{k},\boldsymbol{k}')\left(a_\theta(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger - a_{\theta'}(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger\right).\end{aligned} \tag{471}$$

Also from conditions (466) and (470) we get the following relation for the polarization angles

$$\theta_{21}(\boldsymbol{k}) + \theta_{21}(\boldsymbol{k}') = \theta_{22}(\boldsymbol{k}) + \theta_{22}(\boldsymbol{k}') + \frac{\pi}{2} + n\pi, \qquad n \in \boldsymbol{Z}. \tag{472}$$

We can say about $\Psi_{22}(N)$: both photons have the same polarizations in circular and linear basis. Then the inner product reads

$$\begin{aligned}&\langle O(N)|\Psi_{22}(N)^\dagger\Psi_{22}(N)|O(N)\rangle \\ =\ &\frac{4}{N}\int d\Gamma(\boldsymbol{k})|\psi_{++}(\boldsymbol{k},\boldsymbol{k})|^2 Z(\boldsymbol{k}) + \frac{4(N-1)}{N}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|\psi_{++}(\boldsymbol{k},\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}'),\end{aligned} \tag{473}$$

and for the $N \to \infty$ limit we have

$$\lim_{N\to\infty}\langle O(N)|\Psi_{22}(N)^\dagger\Psi_{22}(N)|O(N)\rangle = 4\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|\psi_{++}(\boldsymbol{k},\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}'). \tag{474}$$

We will refer to operators (456), (460), (467), (471) as the four Bell state corresponding field operators.

6.3 Results and conclusions

This chapter contains new results. As shown in the previous section it is possible to model Bell states in quantum field theory background of N-oscillator reducible representation. The main assumption is that Bell states are maximally correlated or maximally anti-correlated in two polarization bases: circular and linear. However it should be stressed here that in this model the linear polarization angles are dependent on momentum, and from the condition for maximal correlation in both bases we get conditions on the fields and on the polarization angle functions (455), (459), (466) and (470). In the next chapter it will turn out that employing such momentum dependent polarization angle is important for maintaining Lorentz covariance in both bases.

Another point should be mentioned. For the reducible representation the N parameter does not necessary have to go to infinity, since each oscillator is a superposition of already infinitely many different momentum states. It was shown by Wilczewski and Czachor [17], [18], on the example of Rabi oscillations in lossy cavities, that for the reducible representations of N-oscillator, the N parameter should be indeed a very large but finite number. Also the convergence of vacuum energy, shown earlier in section 4.2 for such representation, does not necessary require the N parameter to be infinity. Here it has been shown that when taking limit $N \to \infty$ we lose the (452) term of the inner product that for large finite N could take a small value. Later in chapter 8 it will be shown that this has its consequence in the value of the correlation function for maximally anti-correlated in circular basis photons.

7 Transformation properties of two-photon fields

The main difficulty for Lorentz transformation law of Bell states is the dependence of Wigner rotations $\Theta(\Lambda, \boldsymbol{k})$ on momentum. This was discussed by Peres, Scudo and Terno in paper [57]. They concluded that the spin density matrix for a single spin 1/2 particle is not invariant. When observed form a moving frame, Wigner rotations entangle the spin with the particles momentum distribution. In this sense, in the relativistic context spin and momentum are not independent degrees of freedom. It is not possible to change the inertial reference frame without changing the quantization axis of spin. In consequence Lorentz boosts introduce a transfer of entanglement between the degrees on freedom. This could be useful for entanglement manipulation. Lorentz boost acts as a global transformation on spin and momentum and not as a local transformation strictly on spin or strictly on momentum. The entanglement between spin and momentum alone may not be invariant, the entanglement of the entire field (spin and momentum) is invariant.

Also Ahn, Lee and Hwang [67] studied Lorentz transformation of massive two-particle entangled quantum states. They concluded that to an observer in a moving frame, Bell states appear as rotations or linear combinations of Bell states in that frame.

Furthermore, Czachor [90] investigated relativistic analogues of EPR states for photons and asked if it is possible to find scalar fields that involve maximal entanglement in two bases and in all reference frames. In this chapter we find answers to this question. The chapter is organized as follows. First in section 7.1 we will discuss the effect that the choice of a non-unique vacuum has on Lorentz transformations. Further in section 7.2 transformation properties of a two-photon state will be considered. In section 7.3 and 7.4 the transformation properties of the Bell states are discussed. Here two assumptions are made. Bell states are maximally correlated or anti-correlated in two polarization bases and transform in circular polarization basis under Lorentz transformation as scalars. These two assumptions imply the transformation rule on the polarization angle function and such transformation rule leads to Lorentz invariants also in the linear basis.

7.1 Scalar field

First let us consider an invariant two-photon field operator $\Psi(N)$, i.e.

$$U(\Lambda, 0, N)\Psi(N)U(\Lambda, 0, N)^{\dagger} \quad = \quad \Psi(N). \tag{475}$$

In reducible representation, where vacuum is non-unique, performing a Lorentz transformation on states corresponding to invariant field operators does not result in the same state, since the vacuum also transforms, here as a scalar field (408), so that

$$U(\Lambda, 0, N)|\Psi(N)\rangle \;=\; U(\Lambda, 0, N)\Psi(N)U(\Lambda, 0, N)^{\dagger}U(\Lambda, 0, N)|O(N)\rangle \;=\; \Psi(N)|O_{\Lambda}(N)\rangle. \tag{476}$$

Of course, we make the assumption that the scalar product is conserved under Lorentz transformation, so that

$$\begin{aligned}
& \langle O_{\Lambda}(N)|\Psi(N)^{\dagger}\Psi(N)|O_{\Lambda}(N)\rangle \\
= \;& 2\sum_{s,s'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k'})|\psi_{ss'}(\boldsymbol{k},\boldsymbol{k'})|^2\langle O_{\Lambda}(N)|I(\boldsymbol{k},N)I(\boldsymbol{k'},N)|O_{\Lambda}(N)\rangle \\
= \;& \frac{2}{N}\sum_{s,s'=\pm}\int d\Gamma(\boldsymbol{k})|\psi_{ss'}(\boldsymbol{k},\boldsymbol{k})|^2 Z(\boldsymbol{\Lambda^{-1}k}) \\
+ \;& \frac{2(N-1)}{N}\sum_{s,s'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k'})|\psi_{ss'}(\boldsymbol{k},\boldsymbol{k'})|^2 Z(\boldsymbol{\Lambda^{-1}k})Z(\boldsymbol{\Lambda^{-1}k'}) \\
= \;& \frac{2}{N}\sum_{s,s'=\pm}\int d\Gamma(\boldsymbol{k})|\psi_{ss'}(\boldsymbol{\Lambda k},\boldsymbol{\Lambda k})|^2 Z(\boldsymbol{k}) \\
+ \;& \frac{2(N-1)}{N}\sum_{s,s'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k'})|\psi_{ss'}(\boldsymbol{\Lambda k},\boldsymbol{\Lambda k'})|^2 Z(\boldsymbol{k})Z(\boldsymbol{k'}),
\end{aligned} \tag{477}$$

and this implies the following condition on the field

$$|\psi_{ss'}(\boldsymbol{\Lambda k},\boldsymbol{\Lambda k'})|^2 \quad = \quad |\psi_{ss'}(\boldsymbol{k},\boldsymbol{k'})|^2. \tag{478}$$

7.2 Transformation properties of a two-photon field operator

First let us state that under Lorentz transformation we have the following transformation rule for creation operators in circular basis:

$$U(\Lambda,0,N)^\dagger a_s(\boldsymbol{k},N)^\dagger a_{s'}(\boldsymbol{k}',N)^\dagger U(\Lambda,0,N) = e^{2is\Theta(\Lambda,\boldsymbol{k})}e^{2is'\Theta(\Lambda,\boldsymbol{k}')}a_s(\boldsymbol{\Lambda^{-1}k},N)^\dagger a_{s'}(\boldsymbol{\Lambda^{-1}k'},N)^\dagger, \quad (479)$$

$$U(\Lambda,0,N)a_s(\boldsymbol{k},N)^\dagger a_{s'}(\boldsymbol{k}',N)^\dagger U(\Lambda,0,N)^\dagger = e^{-2is\Theta(\Lambda,\boldsymbol{\Lambda k})}e^{-2is'\Theta(\Lambda,\boldsymbol{\Lambda k}')}a_s(\boldsymbol{\Lambda k},N)^\dagger a_{s'}(\boldsymbol{\Lambda k'},N)^\dagger. \quad (480)$$

Now we will consider a two-photon field operator in circular basis. Let us assume that the field operator $\Psi(N)$ (444) satisfies the scalar field condition, so that under Lorentz transformation we have

$$\begin{aligned}
& U(\Lambda,0,N)\Psi(N)U(\Lambda,0,N)^\dagger \\
= \; & U(\Lambda,0,N)\sum_{s,s'}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \psi_{ss'}(\boldsymbol{k},\boldsymbol{k}')a_s(\boldsymbol{k},N)^\dagger a_{s'}(\boldsymbol{k}',N)^\dagger U(\Lambda,0,N)^\dagger \\
= \; & \sum_{s,s'}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \psi_{ss'}(\boldsymbol{k},\boldsymbol{k}')e^{-2is\Theta(\Lambda,\boldsymbol{\Lambda k})}e^{-2is'\Theta(\Lambda,\boldsymbol{\Lambda k}')}a_s(\boldsymbol{\Lambda k},N)^\dagger a_{s'}(\boldsymbol{\Lambda k'},N)^\dagger \\
= \; & \sum_{s,s'}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \psi_{ss'}(\boldsymbol{\Lambda^{-1}k},\boldsymbol{\Lambda^{-1}k'})e^{-2is\Theta(\Lambda,\boldsymbol{k})}e^{-2is'\Theta(\Lambda,\boldsymbol{k}')}a_s(\boldsymbol{k},N)^\dagger a_{s'}(\boldsymbol{k}',N)^\dagger \\
= \; & \Psi(N). \qquad (481)
\end{aligned}$$

This implies the following transformation rule for the fields:

$$\psi_{ss'}(\boldsymbol{k},\boldsymbol{k'})e^{2is\Theta(\Lambda,\boldsymbol{k})}e^{2is'\Theta(\Lambda,\boldsymbol{k}')} = \psi_{ss'}(\boldsymbol{\Lambda^{-1}k},\boldsymbol{\Lambda^{-1}k'}), \quad (482)$$

which is consistent with (478).

7.3 Transformation properties of states maximally anti-correlated in circular basis

Now let us assume that the field operator $\Psi_{11}(N)$ corresponding to one of the Bell states (456) transforms as a scalar field under Lorentz transformation, so that

$$\begin{aligned}
& U(\Lambda,0,N)\Psi_{11}(N)U(\Lambda,0,N)^\dagger \\
= \; & U(\Lambda,0,N)\sum_{s=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \psi_{s-s}(\boldsymbol{k},\boldsymbol{k}')a_s(\boldsymbol{k},N)^\dagger a_{-s}(\boldsymbol{k}',N)^\dagger U(\Lambda,0,N)^\dagger \\
= \; & \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \psi_{+-}(\boldsymbol{k},\boldsymbol{k}')e^{-2i\Theta(\Lambda,\boldsymbol{\Lambda k})}e^{2i\Theta(\Lambda,\boldsymbol{\Lambda k}')}a_+(\boldsymbol{\Lambda k},N)^\dagger a_-(\boldsymbol{\Lambda k'},N)^\dagger \\
+ \; & \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \psi_{-+}(\boldsymbol{k},\boldsymbol{k}')e^{2i\Theta(\Lambda,\boldsymbol{\Lambda k})}e^{-2i\Theta(\Lambda,\boldsymbol{\Lambda k}')}a_-(\boldsymbol{\Lambda k},N)^\dagger a_+(\boldsymbol{\Lambda k'},N)^\dagger \\
= \; & \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \psi_{+-}(\boldsymbol{\Lambda k},\boldsymbol{\Lambda k'})a_+(\boldsymbol{\Lambda k},N)^\dagger a_-(\boldsymbol{\Lambda k'},N)^\dagger \\
+ \; & \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \psi_{-+}(\boldsymbol{\Lambda k},\boldsymbol{\Lambda k'})a_-(\boldsymbol{\Lambda k},N)^\dagger a_+(\boldsymbol{\Lambda k'},N)^\dagger = \Psi_{11}(N). \qquad (483)
\end{aligned}$$

This implies the following transformation rules for the fields:

$$\psi_{+-}(\boldsymbol{\Lambda^{-1}k},\boldsymbol{\Lambda^{-1}k'}) = \psi_{+-}(\boldsymbol{k},\boldsymbol{k}')e^{2i\Theta(\Lambda,\boldsymbol{k})}e^{-2i\Theta(\Lambda,\boldsymbol{k}')}, \quad (484)$$

$$\psi_{-+}(\boldsymbol{\Lambda^{-1}k},\boldsymbol{\Lambda^{-1}k'}) = \psi_{-+}(\boldsymbol{k},\boldsymbol{k}')e^{-2i\Theta(\Lambda,\boldsymbol{k})}e^{2i\Theta(\Lambda,\boldsymbol{k}')}. \quad (485)$$

On the other hand from the condition for maximally anti-correlated states in linear basis (455) we get

$$\begin{aligned}
\psi_{-+}(\boldsymbol{\Lambda^{-1}k},\boldsymbol{\Lambda^{-1}k'}) &= -e^{2i(\theta_{11}(\boldsymbol{\Lambda^{-1}k})-\theta_{11}(\boldsymbol{\Lambda^{-1}k'}))}\psi_{+-}(\boldsymbol{\Lambda^{-1}k},\boldsymbol{\Lambda^{-1}k'}) \\
&= -e^{2i(\theta_{11}(\boldsymbol{\Lambda^{-1}k})-\theta_{11}(\boldsymbol{\Lambda^{-1}k'}))}\psi_{+-}(\boldsymbol{k},\boldsymbol{k}')e^{2i\Theta(\Lambda,\boldsymbol{k})}e^{-2i\Theta(\Lambda,\boldsymbol{k}')} \\
&= e^{2i(\theta_{11}(\boldsymbol{\Lambda^{-1}k})-\theta_{11}(\boldsymbol{\Lambda^{-1}k'}))}\psi_{-+}(\boldsymbol{k},\boldsymbol{k}')e^{-2i(\theta_{11}(\boldsymbol{k})-\theta_{11}(\boldsymbol{k}'))}e^{2i\Theta(\Lambda,\boldsymbol{k})}e^{-2i\Theta(\Lambda,\boldsymbol{k}')}. \qquad (486)
\end{aligned}$$

Comparison of (485) and (486) implies the transformation rule for the polarization angle under Lorentz transformation:

$$
\begin{aligned}
e^{-2i(\theta_{11}(\mathbf{\Lambda^{-1}}\boldsymbol{k})-\theta_{11}(\mathbf{\Lambda^{-1}}\boldsymbol{k}'))} &= e^{-2i(\theta_{11}(\boldsymbol{k})-\theta_{11}(\boldsymbol{k}'))}e^{4i\Theta(\Lambda,\boldsymbol{k})}e^{-4i\Theta(\Lambda,\boldsymbol{k}')}, &(487)\\
\theta_{11}(\mathbf{\Lambda^{-1}}\boldsymbol{k}) &= \theta_{11}(\boldsymbol{k})-2\Theta(\Lambda,\boldsymbol{k}), &(488)\\
\theta_{11}(\mathbf{\Lambda}\boldsymbol{k}) &= \theta_{11}(\boldsymbol{k})+2\Theta(\Lambda,\mathbf{\Lambda}\boldsymbol{k}). &(489)
\end{aligned}
$$

We can interpret this as if the polarization angle due to Lorentz transformation is shifted by the Wigner phase. With this condition it can be shown that indeed the field operator $\Psi_{11}(N)$ transforms as a scalar in both bases: linear and circular. Let us remind ourselves that the Wigner phase depends only on the direction of momentum, not on the frequency, so that all parallel wave vectors correspond to the same rotational angle. This was shown by Caban and Rembieliński in [70]. Now taking into account conditions (455) and transformation rules for the spin-frames shown in appendix (B.10)-(B.13) we can write an example of the fields and polarization angle in terms of the null tetrad:

$$
\begin{aligned}
\psi_{+-}(\boldsymbol{k},\boldsymbol{k}') &= m_a(\boldsymbol{k})\bar{m}^a(\boldsymbol{k}'), &(490)\\
\psi_{-+}(\boldsymbol{k},\boldsymbol{k}') &= \bar{m}_a(\boldsymbol{k})m^a(\boldsymbol{k}'), &(491)\\
e^{2i(\theta_{11}(\boldsymbol{k})-\theta_{11}(\boldsymbol{k}'))} &= -\frac{\bar{m}_a(\boldsymbol{k})m^a(\boldsymbol{k}')}{m_b(\boldsymbol{k})\bar{m}^b(\boldsymbol{k}')}. &(492)
\end{aligned}
$$

We can also write the field operator $\Psi_{11}(N)$ with respect to the null tetrad, i.e.

$$
\begin{aligned}
\Psi_{11}(N) &= \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \left(m_a(\boldsymbol{k})\bar{m}^a(\boldsymbol{k}')a_+(\boldsymbol{k},N)^\dagger a_-(\boldsymbol{k}',N)^\dagger+\bar{m}_a(\boldsymbol{k})m^a(\boldsymbol{k}')a_-(\boldsymbol{k},N)^\dagger a_+(\boldsymbol{k}',N)^\dagger\right)\\
&= 2i\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{-i(\theta_{11}(\boldsymbol{k})-\theta_{11}(\boldsymbol{k}'))}\bar{m}_a(\boldsymbol{k})m^a(\boldsymbol{k}')a_\theta(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger. \qquad (493)
\end{aligned}
$$

Under Lorentz transformation we have the following transformation rules for creation operators in linear basis

$$
U(\Lambda,0,N)a_\theta(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger U(\Lambda,0,N)^\dagger = a_\theta(\mathbf{\Lambda}\boldsymbol{k},N)^\dagger a_{\theta'}(\mathbf{\Lambda}\boldsymbol{k}',N)^\dagger, \qquad (494)
$$

$$
U(\Lambda,0,N)a_\theta(\boldsymbol{k},N)^\dagger a_{\theta}(\boldsymbol{k}',N)^\dagger U(\Lambda,0,N)^\dagger = a_\theta(\mathbf{\Lambda}\boldsymbol{k},N)^\dagger a_{\theta}(\mathbf{\Lambda}\boldsymbol{k}',N)^\dagger, \qquad (495)
$$

$$
U(\Lambda,0,N)a_{\theta'}(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger U(\Lambda,0,N)^\dagger = a_{\theta'}(\mathbf{\Lambda}\boldsymbol{k},N)^\dagger a_{\theta'}(\mathbf{\Lambda}\boldsymbol{k}',N)^\dagger. \qquad (496)
$$

This is derived step by step in appendix (H.10) - (H.12) and it should be stressed that, for this calculus, the transformation rule for the polarization angle was taken into account. Now let us see how the field operator $\Psi_{11}(N)$ in linear basis transforms under Lorentz transformation. Using the transformation formula (494) we find that

$$
\begin{aligned}
&U(\Lambda,0,N)\Psi_{11}(N)U(\Lambda,0,N)^\dagger\\
&= 2i\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{-i(\theta_{11}(\boldsymbol{k})-\theta_{11}(\boldsymbol{k}'))}\bar{m}_a(\boldsymbol{k})m^a(\boldsymbol{k}')a_\theta(\mathbf{\Lambda}\boldsymbol{k},N)^\dagger a_{\theta'}(\mathbf{\Lambda}\boldsymbol{k}',N)^\dagger\\
&= 2i\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{-i(\theta_{11}(\mathbf{\Lambda^{-1}}\boldsymbol{k})-\theta_{11}(\mathbf{\Lambda^{-1}}\boldsymbol{k}'))}\bar{m}_a(\mathbf{\Lambda^{-1}}\boldsymbol{k})m^a(\mathbf{\Lambda^{-1}}\boldsymbol{k}')a_\theta(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger\\
&= 2i\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{-i(\theta_{11}(\boldsymbol{k})-\theta_{11}(\boldsymbol{k}'))}\bar{m}_a(\boldsymbol{k})m^a(\boldsymbol{k}')a_\theta(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger. \qquad (497)
\end{aligned}
$$

As one can see such a state is still maximally correlated after Lorentz transformation. This was discussed earlier by M. Czachor in [90] and by H. Terashima and M. Ueda in [60]. The correlation in both bases depends on the relation between the polarization angle $\theta(\boldsymbol{k})$ and the fields. Conclusion is that to maintain maximal entanglement in both bases under Lorentz transformations in EPR-type experiments, one has to employ momentum dependent polarization functions $\theta(\boldsymbol{k})$ that compensate the Wigner phase $2\Theta(\Lambda,\boldsymbol{k})$. Now let us consider the field operator $\Psi_{12}(N)$ (460). In circular basis we want this field operator to transform under Lorentz transformation as a scalar:

$$\begin{aligned}
& U(\Lambda,0,N)\Psi_{12}(N)U(\Lambda,0,N)^\dagger \\
= \;& U(\Lambda,0,N)\sum_{s=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k'})\ \psi_{s-s}(\boldsymbol{k},\boldsymbol{k'})a_s(\boldsymbol{k},N)^\dagger a_{-s}(\boldsymbol{k'},N)^\dagger U(\Lambda,0,N)^\dagger \\
= \;& \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k'})\ \psi_{+-}(\boldsymbol{k},\boldsymbol{k'})e^{-2i\Theta(\Lambda,\boldsymbol{\Lambda k})}e^{2i\Theta(\Lambda,\boldsymbol{\Lambda k'})}a_+(\boldsymbol{\Lambda k},N)^\dagger a_-(\boldsymbol{\Lambda k'},N)^\dagger \\
+ \;& \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k'})\ \psi_{-+}(\boldsymbol{k},\boldsymbol{k'})e^{2i\Theta(\Lambda,\boldsymbol{\Lambda k})}e^{-2i\Theta(\Lambda,\boldsymbol{\Lambda k'})}a_-(\boldsymbol{\Lambda k},N)^\dagger a_+(\boldsymbol{\Lambda k'},N)^\dagger \\
= \;& \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k'})\ \psi_{+-}(\boldsymbol{\Lambda k},\boldsymbol{\Lambda k'})a_+(\boldsymbol{\Lambda k},N)^\dagger a_-(\boldsymbol{\Lambda k'},N)^\dagger \\
+ \;& \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k'})\ \psi_{-+}(\boldsymbol{\Lambda k},\boldsymbol{\Lambda k'})a_-(\boldsymbol{\Lambda k},N)^\dagger a_+(\boldsymbol{\Lambda k'},N)^\dagger \\
= \;& \Psi_{12}(N).
\end{aligned} \tag{498}$$

This implies the following transformation rules for the fields:

$$\psi_{+-}(\boldsymbol{k},\boldsymbol{k'})e^{2i\Theta(\Lambda,\boldsymbol{k})}e^{-2i\Theta(\Lambda,\boldsymbol{k'})} = \psi_{+-}(\boldsymbol{\Lambda^{-1}k},\boldsymbol{\Lambda^{-1}k'}), \tag{499}$$

$$\psi_{-+}(\boldsymbol{k},\boldsymbol{k'})e^{-2i\Theta(\Lambda,\boldsymbol{k})}e^{2i\Theta(\Lambda,\boldsymbol{k'})} = \psi_{-+}(\boldsymbol{\Lambda^{-1}k},\boldsymbol{\Lambda^{-1}k'}). \tag{500}$$

Furthermore, from the condition on the polarization angle and the field (459) we get

$$\begin{aligned}
\psi_{-+}(\boldsymbol{\Lambda^{-1}k},\boldsymbol{\Lambda^{-1}k'}) &= e^{2i(\theta_{12}(\boldsymbol{\Lambda^{-1}k})-\theta_{12}(\boldsymbol{\Lambda^{-1}k'}))}\psi_{+-}(\boldsymbol{\Lambda^{-1}k},\boldsymbol{\Lambda^{-1}k'}) \\
&= e^{2i(\theta_{12}(\boldsymbol{\Lambda^{-1}k})-\theta_{12}(\boldsymbol{\Lambda^{-1}k'}))}\psi_{+-}(\boldsymbol{k},\boldsymbol{k'})e^{2i\Theta(\Lambda,\boldsymbol{k})}e^{-2i\Theta(\Lambda,\boldsymbol{k'})} \\
&= e^{2i(\theta_{12}(\boldsymbol{\Lambda^{-1}k})-\theta_{12}(\boldsymbol{\Lambda^{-1}k'}))}\psi_{-+}(\boldsymbol{k},\boldsymbol{k'})e^{-2i(\theta_{12}(\boldsymbol{k})-\theta_{12}(\boldsymbol{k'}))}e^{2i\Theta(\Lambda,\boldsymbol{k})}e^{-2i\Theta(\Lambda,\boldsymbol{k'})},
\end{aligned} \tag{501}$$

and this implies the transformation rule for the polarization angle $\theta_{12}(\boldsymbol{k})$:

$$e^{-2i(\theta_{12}(\boldsymbol{\Lambda^{-1}k})-\theta_{12}(\boldsymbol{\Lambda^{-1}k'}))} = e^{-2i(\theta_{12}(\boldsymbol{k})-\theta_{12}(\boldsymbol{k'}))}e^{4i\Theta(\Lambda,\boldsymbol{k})}e^{-4i\Theta(\Lambda,\boldsymbol{k'})}, \tag{502}$$

$$\theta_{12}(\boldsymbol{\Lambda^{-1}k}) = \theta_{12}(\boldsymbol{k}) - 2\Theta(\Lambda,\boldsymbol{k}). \tag{503}$$

Taking into account conditions (459) and the transformation rules on the spin-frames (B.10) - (B.13), we can write an example of the fields and polarization angle in terms of the null tetrad:

$$\psi_{+-}(\boldsymbol{k},\boldsymbol{k'}) = m_a(\boldsymbol{k})\bar{m}^a(\boldsymbol{k'}), \tag{504}$$

$$\psi_{-+}(\boldsymbol{k},\boldsymbol{k'}) = \bar{m}_a(\boldsymbol{k})m^a(\boldsymbol{k'}), \tag{505}$$

$$e^{2i(\theta_{12}(\boldsymbol{k})-\theta_{12}(\boldsymbol{k'}))} = \frac{\bar{m}_a(\boldsymbol{k})m^a(\boldsymbol{k'})}{m_b(\boldsymbol{k})\bar{m}^b(\boldsymbol{k'})}. \tag{506}$$

Now we can write the field operator $\Psi_{12}(N)$ (460) with respect to the null tetrad

$$\Psi_{12}(N) = \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k'})\ e^{i(\theta_{12}(\boldsymbol{k})-\theta_{12}(\boldsymbol{k'}))}m_a(\boldsymbol{k})\bar{m}^a(\boldsymbol{k'})\left(a_\theta(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k'},N)^\dagger + a_{\theta'}(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k'},N)^\dagger\right). \tag{507}$$

Under Lorentz transformation this field operator in linear basis transforms, due to the transformation rule on the polarization angle (503), as a scalar, i.e.

$$
\begin{aligned}
& U(\Lambda,0,N)\Psi_{12}(N)U(\Lambda,0,N)^\dagger \\
= & \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{i(\theta_{12}(\boldsymbol{k})-\theta_{12}(\boldsymbol{k}'))}m_a(\boldsymbol{k})\bar{m}^a(\boldsymbol{k}')\left(a_\theta(\boldsymbol{\Lambda k},N)^\dagger a_\theta(\boldsymbol{\Lambda k}',N)^\dagger + a_{\theta'}(\boldsymbol{\Lambda k},N)^\dagger a_{\theta'}(\boldsymbol{\Lambda k}',N)^\dagger\right) \\
= & \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{i(\theta_{12}(\boldsymbol{\Lambda^{-1}k})-\theta_{12}(\boldsymbol{\Lambda^{-1}k}'))}m_a(\boldsymbol{\Lambda^{-1}k})\bar{m}^a(\boldsymbol{\Lambda^{-1}k}') \\
\times & \left(a_\theta(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger + a_{\theta'}(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger\right) \\
= & \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{i(\theta_{12}(\boldsymbol{\Lambda^{-1}k})-\theta_{12}(\boldsymbol{\Lambda^{-1}k}'))}e^{2i\Theta(\Lambda,\boldsymbol{k})}e^{-2i\Theta(\Lambda,\boldsymbol{k}')}m_a(\boldsymbol{k})\bar{m}^a(\boldsymbol{k}') \\
\times & \left(a_\theta(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger + a_{\theta'}(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger\right) \\
= & \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{i(\theta_{12}(\boldsymbol{k})-\theta_{12}(\boldsymbol{k}'))}m_a(\boldsymbol{k})\bar{m}^a(\boldsymbol{k}')\left(a_\theta(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger + a_{\theta'}(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger\right) \\
= & \Psi_{12}(N). \qquad (508)
\end{aligned}
$$

7.4 Transformation properties of states maximally correlated in circular basis

For the field operator $\Psi_{21}(N)$ (467) we first assume the following transformation rule in circular basis

$$
\begin{aligned}
& U(\Lambda,0,N)\Psi_{21}(N)U(\Lambda,0,N)^\dagger \\
= & U(\Lambda,0,N)\sum_{s=\mp}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \psi_{ss}(\boldsymbol{k},\boldsymbol{k}')a_s(\boldsymbol{k},N)^\dagger a_s(\boldsymbol{k}',N)^\dagger U(\Lambda,0,N)^\dagger \\
= & \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \psi_{++}(\boldsymbol{k},\boldsymbol{k}')e^{2i\Theta(\Lambda,\boldsymbol{\Lambda k})}e^{2i\Theta(\Lambda,\boldsymbol{\Lambda k}')}a_+(\boldsymbol{\Lambda k},N)^\dagger a_+(\boldsymbol{\Lambda k}',N)^\dagger \\
+ & \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \psi_{--}(\boldsymbol{k},\boldsymbol{k}')e^{-2i\Theta(\Lambda,\boldsymbol{\Lambda k})}e^{-2i\Theta(\Lambda,\boldsymbol{\Lambda k}')}a_-(\boldsymbol{\Lambda k},N)^\dagger a_-(\boldsymbol{\Lambda k}',N)^\dagger \\
= & \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \psi_{++}(\boldsymbol{\Lambda k},\boldsymbol{\Lambda k}')a_+(\boldsymbol{\Lambda k},N)^\dagger a_+(\boldsymbol{\Lambda k}',N)^\dagger \\
+ & \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \psi_{--}(\boldsymbol{\Lambda k},\boldsymbol{\Lambda k}')a_-(\boldsymbol{\Lambda k},N)^\dagger a_-(\boldsymbol{\Lambda k}',N)^\dagger \ = \ \Psi_{21}(N). \qquad (509)
\end{aligned}
$$

This implies that the fields have to transform as:

$$
\begin{aligned}
\psi_{++}(\boldsymbol{k},\boldsymbol{k}')e^{2i\Theta(\Lambda,\boldsymbol{k})}e^{2i\Theta(\Lambda,\boldsymbol{k}')} &= \psi_{++}(\boldsymbol{\Lambda^{-1}k},\boldsymbol{\Lambda^{-1}k}'), \qquad (510)\\
\psi_{--}(\boldsymbol{k},\boldsymbol{k}')e^{-2i\Theta(\Lambda,\boldsymbol{k})}e^{-2i\Theta(\Lambda,\boldsymbol{k}')} &= \psi_{--}(\boldsymbol{\Lambda^{-1}k},\boldsymbol{\Lambda^{-1}k}'). \qquad (511)
\end{aligned}
$$

From (466) we get

$$
\begin{aligned}
\psi_{--}(\boldsymbol{\Lambda^{-1}k},\boldsymbol{\Lambda^{-1}k}') &= -e^{2i(\theta_{21}(\boldsymbol{\Lambda^{-1}k})+\theta_{21}(\boldsymbol{\Lambda^{-1}k}'))}\psi_{++}(\boldsymbol{\Lambda^{-1}k},\boldsymbol{\Lambda^{-1}k}') \\
&= -e^{2i(\theta_{21}(\boldsymbol{\Lambda^{-1}k})-\theta_{21}(\boldsymbol{\Lambda^{-1}k}'))}\psi_{++}(\boldsymbol{k},\boldsymbol{k}')e^{2i\Theta(\Lambda,\boldsymbol{k})}e^{2i\Theta(\Lambda,\boldsymbol{k}')} \\
&= e^{2i(\theta_{21}(\boldsymbol{\Lambda^{-1}k})+\theta_{21}(\boldsymbol{\Lambda^{-1}k}'))}\psi_{--}(\boldsymbol{k},\boldsymbol{k}')e^{-2i(\theta_{21}(\boldsymbol{k})+\theta_{21}(\boldsymbol{k}'))}e^{2i\Theta(\Lambda,\boldsymbol{k})}e^{2i\Theta(\Lambda,\boldsymbol{k}')}.
\end{aligned}
\qquad (512)
$$

This implies the following transformation rule for the polarization angle $\theta_{21}(\boldsymbol{k})$:

$$
\begin{aligned}
e^{2i(\theta_{21}(\boldsymbol{\Lambda^{-1}k})+\theta_{21}(\boldsymbol{\Lambda^{-1}k}'))} &= e^{2i(\theta_{21}(\boldsymbol{k})+\theta_{21}(\boldsymbol{k}'))}e^{-4i\Theta(\Lambda,\boldsymbol{k})}e^{-4i\Theta(\Lambda,\boldsymbol{k}')}, \qquad (513)\\
\theta_{21}(\boldsymbol{\Lambda^{-1}k}) &= \theta_{21}(\boldsymbol{k})-2\Theta(\Lambda,\boldsymbol{k}). \qquad (514)
\end{aligned}
$$

Taking into account conditions (466) and transformation rules for the spin-frames (B.10) - (B.13), we can write an example of fields and linear polarization functions in terms of the null tetrad:

$$
\begin{aligned}
\psi_{++}(\boldsymbol{k},\boldsymbol{k}') &= m_a(\boldsymbol{k})m^a(\boldsymbol{k}'), \qquad (515)\\
\psi_{--}(\boldsymbol{k},\boldsymbol{k}') &= \bar{m}_a(\boldsymbol{k})\bar{m}^a(\boldsymbol{k}'), \qquad (516)\\
e^{2i(\theta_{21}(\boldsymbol{k})+\theta_{21}(\boldsymbol{k}'))} &= -\frac{\bar{m}_a(\boldsymbol{k})\bar{m}^a(\boldsymbol{k}')}{m_b(\boldsymbol{k})m^b(\boldsymbol{k}')}. \qquad (517)
\end{aligned}
$$

Now we can write the field operator $\Psi_{21}(N)$ (467) with respect to the null tetrad in circular and linear bases

$$\begin{aligned}\Psi_{21}(N) &= \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \left(m_a(\boldsymbol{k})m^a(\boldsymbol{k}')a_+(\boldsymbol{k},N)^\dagger a_+(\boldsymbol{k}',N)^\dagger + \bar{m}_a(\boldsymbol{k})\bar{m}^a(\boldsymbol{k}')a_-(\boldsymbol{k},N)^\dagger a_-(\boldsymbol{k}',N)^\dagger\right) \\ &= -2i\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{-i(\theta_{21}(\boldsymbol{k})+\theta_{21}(\boldsymbol{k}'))}\bar{m}_a(\boldsymbol{k})\bar{m}^a(\boldsymbol{k}')a_\theta(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger. \end{aligned} \tag{518}$$

Let us also see how the field operator $\Psi_{21}(N)$ (467) in linear basis transforms under Lorentz transformation. Using the transformation law (494) we find

$$\begin{aligned} & U(\Lambda,0,N)\Psi_{21}(N)U(\Lambda,0,N)^\dagger \\ = & -i\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{-i(\theta_{21}(\boldsymbol{k})+\theta_{21}(\boldsymbol{k}'))}\bar{m}_a(\boldsymbol{k})\bar{m}^a(\boldsymbol{k}')a_\theta(\boldsymbol{\Lambda k},N)^\dagger a_{\theta'}(\boldsymbol{\Lambda k}',N)^\dagger \\ = & -i\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{-i(\theta_{21}(\boldsymbol{\Lambda^{-1}k})+\theta_{21}(\boldsymbol{\Lambda^{-1}k}'))}\bar{m}_a(\boldsymbol{\Lambda^{-1}k})\bar{m}^a(\boldsymbol{\Lambda^{-1}k}')a_\theta(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger \\ = & -i\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{-i(\theta_{21}(\boldsymbol{\Lambda^{-1}k})+\theta_{21}(\boldsymbol{\Lambda^{-1}k}'))}e^{-2i\Theta(\Lambda,\boldsymbol{k})}e^{-2i\Theta(\Lambda,\boldsymbol{k}')}\bar{m}_a(\boldsymbol{k})\bar{m}^a(\boldsymbol{k}')a_\theta(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger \\ = & -i\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{-i(\theta_{21}(\boldsymbol{k})+\theta_{21}(\boldsymbol{k}'))}\bar{m}_a(\boldsymbol{k})\bar{m}^a(\boldsymbol{k}')a_\theta(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger \\ = & \Psi_{21}(N). \end{aligned} \tag{519}$$

Finally we will consider the field operator corresponding to the field operator $\Psi_{22}(N)$ (471). First we would want this operator in circular basis to transform as a scalar field:

$$\begin{aligned} & U(\Lambda,0,N)\Psi_{22}(N)U(\Lambda,0,N)^\dagger \\ = & U(\Lambda,0,N)\sum_{s=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \psi_{ss}(\boldsymbol{k},\boldsymbol{k}')a_s(\boldsymbol{k},N)^\dagger a_s(\boldsymbol{k}',N)^\dagger U(\Lambda,0,N)^\dagger \\ = & \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \psi_{++}(\boldsymbol{k},\boldsymbol{k}')e^{2i\Theta(\Lambda,\boldsymbol{\Lambda k})}e^{2i\Theta(\Lambda,\boldsymbol{\Lambda k}')}a_+(\boldsymbol{\Lambda k},N)^\dagger a_+(\boldsymbol{\Lambda k}',N)^\dagger \\ + & \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \psi_{--}(\boldsymbol{k},\boldsymbol{k}')e^{-2i\Theta(\Lambda,\boldsymbol{\Lambda k})}e^{-2i\Theta(\Lambda,\boldsymbol{\Lambda k}')}a_-(\boldsymbol{\Lambda k},N)^\dagger a_-(\boldsymbol{\Lambda k}',N)^\dagger \\ = & \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \psi_{++}(\boldsymbol{\Lambda k},\boldsymbol{\Lambda k}')a_+(\boldsymbol{\Lambda k},N)^\dagger a_+(\boldsymbol{\Lambda k}',N)^\dagger \\ + & \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \psi_{--}(\boldsymbol{\Lambda k},\boldsymbol{\Lambda k}')a_-(\boldsymbol{\Lambda k},N)^\dagger a_-(\boldsymbol{\Lambda k}',N)^\dagger = \Psi_{22}(N). \end{aligned} \tag{520}$$

This implies the following transformation rule on the field

$$\begin{aligned}\psi_{++}(\boldsymbol{k},\boldsymbol{k}')e^{2i\Theta(\Lambda,\boldsymbol{k})}e^{2i\Theta(\Lambda,\boldsymbol{k}')} &= \psi_{++}(\boldsymbol{\Lambda^{-1}k},\boldsymbol{\Lambda^{-1}k}'), \\ \psi_{--}(\boldsymbol{k},\boldsymbol{k}')e^{-2i\Theta(\Lambda,\boldsymbol{k})}e^{-2i\Theta(\Lambda,\boldsymbol{k}')} &= \psi_{--}(\boldsymbol{\Lambda^{-1}k},\boldsymbol{\Lambda^{-1}k}'). \end{aligned} \tag{521}$$

From (470) we get

$$\begin{aligned}\psi_{--}(\boldsymbol{\Lambda^{-1}k},\boldsymbol{\Lambda^{-1}k}') &= e^{2i(\theta_{22}(\boldsymbol{\Lambda^{-1}k})+\theta_{22}(\boldsymbol{\Lambda^{-1}k}'))}\psi_{++}(\boldsymbol{\Lambda^{-1}k},\boldsymbol{\Lambda^{-1}k}') \\ &= e^{2i(\theta_{22}(\boldsymbol{\Lambda^{-1}k})-\theta_{22}(\boldsymbol{\Lambda^{-1}k}'))}\psi_{++}(\boldsymbol{k},\boldsymbol{k}')e^{2i\Theta(\Lambda,\boldsymbol{k})}e^{2i\Theta(\Lambda,\boldsymbol{k}')} \\ &= e^{2i(\theta_{22}(\boldsymbol{\Lambda^{-1}k})+\theta_{22}(\boldsymbol{\Lambda^{-1}k}'))}\psi_{--}(\boldsymbol{k},\boldsymbol{k}')e^{-2i(\theta_{22}(\boldsymbol{k})+\theta_{22}(\boldsymbol{k}'))}e^{2i\Theta(\Lambda,\boldsymbol{k})}e^{2i\Theta(\Lambda,\boldsymbol{k}')}. \end{aligned} \tag{522}$$

This implies the following transformation rule for the polarization angle $\theta_{22}(\boldsymbol{k})$:

$$e^{2i(\theta_{22}(\boldsymbol{\Lambda^{-1}k})+\theta_{22}(\boldsymbol{\Lambda^{-1}k}'))} = e^{2i(\theta_{22}(\boldsymbol{k})+\theta_{22}(\boldsymbol{k}'))}e^{-4i\Theta(\Lambda,\boldsymbol{k})}e^{-4i\Theta(\Lambda,\boldsymbol{k}')}, \tag{523}$$

$$\theta_{22}(\boldsymbol{\Lambda^{-1}k}) = \theta_{22}(\boldsymbol{k}) - 2\Theta(\Lambda,\boldsymbol{k}). \tag{524}$$

Taking into account condition (470) and transformation rules on the spin-frames (B.10) - (B.13) we can write an example of fields:

$$\begin{aligned}\psi_{++}(\boldsymbol{k},\boldsymbol{k}') &= m_a(\boldsymbol{k})m^a(\boldsymbol{k}'), &(525)\\ \psi_{--}(\boldsymbol{k},\boldsymbol{k}') &= \bar{m}_a(\boldsymbol{k})\bar{m}^a(\boldsymbol{k}'), &(526)\\ e^{2i(\theta_{22}(\boldsymbol{k})+\theta_{22}(\boldsymbol{k}'))} &= \frac{\bar{m}_a(\boldsymbol{k})\bar{m}^a(\boldsymbol{k}')}{m_b(\boldsymbol{k})m^b(\boldsymbol{k}')}. &(527)\end{aligned}$$

Now we can write the field operator $\Psi_{22}(N)$ (471) with respect to the null tetrad

$$\begin{aligned}\Psi_{22}(N) &= \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ \left(m_a(\boldsymbol{k})m^a(\boldsymbol{k}')a_+(\boldsymbol{k},N)^\dagger a_+(\boldsymbol{k}',N)^\dagger + \bar{m}_a(\boldsymbol{k})\bar{m}^a(\boldsymbol{k}')a_-(\boldsymbol{k},N)^\dagger a_-(\boldsymbol{k}',N)^\dagger\right)\\ &= \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{i(\theta_{22}(\boldsymbol{k})+\theta_{22}(\boldsymbol{k}'))}m_a(\boldsymbol{k})m^a(\boldsymbol{k}')\left(a_\theta(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger - a_{\theta'}(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger\right)\\ &= \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{-i(\theta_{22}(\boldsymbol{k})+\theta_{22}(\boldsymbol{k}'))}\bar{m}_a(\boldsymbol{k})\bar{m}^a(\boldsymbol{k}')\left(a_\theta(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger - a_{\theta'}(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger\right).\end{aligned} \tag{528}$$

Under Lorentz transformation such operator also maintains maximal correlations in linear polarization basis:

$$\begin{aligned}& U(\Lambda,0,N)\Psi_{22}(N)U(\Lambda,0,N)^\dagger\\ =& \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{i(\theta_{22}(\boldsymbol{k})+\theta_{22}(\boldsymbol{k}'))}m_a(\boldsymbol{k})m^a(\boldsymbol{k}')\left(a_\theta(\boldsymbol{\Lambda k},N)^\dagger a_\theta(\boldsymbol{\Lambda k}',N)^\dagger - a_{\theta'}(\boldsymbol{\Lambda k},N)^\dagger a_{\theta'}(\boldsymbol{\Lambda k}',N)^\dagger\right)\\ =& \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{i(\theta_{22}(\boldsymbol{\Lambda^{-1}k})+\theta_{22}(\boldsymbol{\Lambda^{-1}k}'))}m_a(\boldsymbol{\Lambda^{-1}k})m^a(\boldsymbol{\Lambda^{-1}k}')\\ \times& \left(a_\theta(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger - a_{\theta'}(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger\right)\\ =& \int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{i(\theta_{22}(\boldsymbol{k})+\theta_{22}(\boldsymbol{k}'))}m_a(\boldsymbol{k})m^a(\boldsymbol{k}')\left(a_\theta(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger - a_{\theta'}(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger\right)\\ =& \Psi_{22}(N).\end{aligned} \tag{529}$$

Without the transformation rule on the polarization angle such states no longer would have maintained the maximal correlation.

7.5 Results and conclusions

This chapter contains new results. It is shown that theoretically it is possible to maintain Lorentz covariance of the field operators corresponding to the four photon Bell states introduced in previous chapter in both polarization bases. The conclusion is: to obtain maximal correlation for EPR-type experiments in both bases one has to employ momentum dependent polarization functions that transform under Lorentz transformation in such a way that they compensate the Wigner phase $2\Theta(\Lambda,\boldsymbol{k})$.

8 Observables in EPR experiment

Since quantum mechanics was born the concepts of its foundations were several times widely discussed and are still till now. One of the triggers for such discussions was the well known Gedankenexperiment proposed in 1935 by Albert Einstein, Boris Podolsky and Nathan Rosen, known as the EPR experiment. In their paper [39], Einstein, Podolsky and Rosen define elements of physical reality as physical quantities, the values of which can be predicted with certainty without in any way disturbing the system. They assume that every element of physical reality needs to have a counterpart in a complete physical theory. In their thought experiment they consider two systems which interact at some time t after which there should be no further interaction. Depending on a measurement of position or momentum on system one, due to the reduction of the wave packet both momentum and position of system two could become an element of reality by their definition. Quantum Mechanics permitted the existence of two-particle states such that one could predict strong correlations both in velocity and position even in case when the particles where widely separated and no longer could interact. But since the operators for momentum and position do not commute, they do not both have a simultaneous counterpart in quantum mechanics. Thus, they concluded quantum mechanics to be incompatible with the local and realistic description. In 1935 the EPR paper, apart from Schrödinger and Bohr, was rather ignored in most debates.

After almost 30 years a short article by John Bell [43] changed this situation and took the EPR arguments very seriously. In his paper Bell introduced so called hidden variables that are given to the two particles at their initial preparation in an entangled state, and carried along by each particle after separation.

It was first shown experimentally by Freedman and Clauser in 1972 [47] that Bell's inequality is violated in the way that quantum theory predicts. In the experiment proposed earlier by Clauser, Horne, Shimony, and Holt (CHSH) [46] they measured linear polarization correlation of photons emitted in an atomic cascade of calcium. It was shown by a generalization of Bell's inequality that the existence of local hidden variables imposes restrictions on this correlation in conflict with the predictions of quantum mechanics, providing strong evidence against local hidden-variable theories.

In 1982 with more advanced equipment Aspect, Grangier and Roger [49] repeated Freedman and Clauser's experiment with far more accurate precision. Two entangled photons were produced in the decay of an excited calcium atom, and each photon was directed by a switch to one of two polarization analyzers, chosen pseudo-randomly. The photons were detected about 12 m apart, corresponding to a light travel time of about 40 ns. This time was considerably longer than either the cycle time of the switch, or the difference in the times of arrival of the two-photons. Therefore the "decision" about which observable to measure was made after the photons were already in flight, and the events that selected the axes for the measurement of photons A and B were space-like separated. The results were consistent with the quantum predictions, and violated the CHSH inequality by five standard deviations. Since Aspect, many other experiments have confirmed this finding.

Furthermore, Ahn et al. [67], [68] also calculated the same situation with all the Bell states and concluded that the Wigner rotation could cause "a counter example for the nonlocality of the EPR paradox".

The main purpose of this chapter is to introduce detection in EPR-type experiment in the background of N-oscillator reducible representations field theory. First in section 8.1 a yes-no observable, for describing measurement on detectors, is introduced. Further in section 8.2 a correlation function for a two-photon state is calculated. In sections 8.3 and 8.4 a correlation function for Bell states is calculated for maximally correlated and anti-correlated in circular basis respectively.

8.1 Yes-no observable

Let us first define a yes-no observable for the linear polarizations:

$$Y_\alpha(\boldsymbol{l}, N) = n_\alpha(\boldsymbol{l}, N) - n_{\alpha'}(\boldsymbol{l}, N). \tag{530}$$

Here $n_\alpha(\boldsymbol{l}, N)$ is the number operator for α oriented polarizations in reducible representations of N-oscillator. In (530) we use the definition (89) for the number operator. This observable may describe measurement in detectors oriented in α direction in EPR-type experiments, and α' is denoted here as

$\alpha' = \alpha + \frac{\pi}{2}$. For later purpose this is rewritten also in circular basis:

$$\begin{aligned} n_\alpha(\boldsymbol{l}, N) &= \sum_{n=1}^{N} \left(|\boldsymbol{l}\rangle\langle\boldsymbol{l}| \otimes a^\dagger_\alpha a_\alpha\right)^{(n)} \\ &= \frac{1}{2}\sum_{n=1}^{N} \Big(|\boldsymbol{l}\rangle\langle\boldsymbol{l}| \otimes a^\dagger_+ a_+ + e^{2i\alpha}|\boldsymbol{l}\rangle\langle\boldsymbol{l}| \otimes a^\dagger_- a_+ + e^{-2i\alpha}|\boldsymbol{l}\rangle\langle\boldsymbol{l}| \otimes a^\dagger_+ a_- + |\boldsymbol{l}\rangle\langle\boldsymbol{l}| \otimes a^\dagger_- a_-\Big)^{(n)}_{,} \end{aligned} \tag{531}$$

$$\begin{aligned} n_{\alpha'}(\boldsymbol{l}, N) &= \sum_{n=1}^{N} \Big(|\boldsymbol{l}\rangle\langle\boldsymbol{l}| \otimes a^\dagger_{\alpha'} a_{\alpha'}\Big)^{(n)} \\ &= \frac{1}{2}\sum_{n=1}^{N} \Big(|\boldsymbol{l}\rangle\langle\boldsymbol{l}| \otimes a^\dagger_+ a_+ - e^{2i\alpha}|\boldsymbol{l}\rangle\langle\boldsymbol{l}| \otimes a^\dagger_- a_+ - e^{-2i\alpha}|\boldsymbol{l}\rangle\langle\boldsymbol{l}| \otimes a^\dagger_+ a_- + |\boldsymbol{l}\rangle\langle\boldsymbol{l}| \otimes a^\dagger_- a_-\Big)^{(n)}_{.} \end{aligned} \tag{532}$$

Now we can write the yes-no observable also with respect to circular polarizations

$$\begin{aligned} Y_\alpha(\boldsymbol{l}, N) &= n_\alpha(\boldsymbol{l}, N) - n_{\alpha'}(\boldsymbol{l}, N) = \sum_{n=1}^{N} \Big(e^{2i\alpha}|\boldsymbol{l}\rangle\langle\boldsymbol{l}| \otimes a^\dagger_- a_+ + e^{-2i\alpha}|\boldsymbol{l}\rangle\langle\boldsymbol{l}| \otimes a^\dagger_+ a_-\Big)^{(n)} \\ &= \sum_{n=1}^{N}\sum_{s=\pm} \Big(e^{2is\alpha}|\boldsymbol{l}\rangle\langle\boldsymbol{l}| \otimes a^\dagger_{-s} a_s\Big)^{(n)}_{.} \end{aligned} \tag{533}$$

Let us note that observable so defined measures the polarization angle with respect to the circular polarization basis (left-handed or right-handed). This can be seen if in (533) we change the sing of the α angle, i.e.

$$Y_{-\alpha}(\boldsymbol{l}, N) = \sum_{n=1}^{N}\sum_{s=\pm} \left(e^{2is\alpha}|\boldsymbol{l}\rangle\langle\boldsymbol{l}| \otimes a^\dagger_s a_{-s}\right)^{(n)}_{.} \tag{534}$$

This remark is important for further interpretation of the correlation functions, where in next sections, for anti-correlated in circular basis states, we get a $\cos(\beta - \alpha)$ term and for correlated in circular basis states a $\cos(\beta + \alpha)$ term. Now for a more realistic case the localization of the photon detector leads to a momentum solid angle spread $\boldsymbol{l} \in \Omega$, and this is why we will consider the following observable

$$Y_\alpha(N) = \int_\Omega d\Gamma(\boldsymbol{l})\ Y_\alpha(\boldsymbol{l}, N). \tag{535}$$

It should be stressed here that the α angle in detectors is fixed for all momentum values of the photon field. This is important for the relativistic background, because it is assumed here that the polarization angle for linearly polarized fields depends on the momentum and is shifted due to the Wigner phase under Lorentz transformation. Let us remind ourselves that this dependence on momentum is necessary for maintaining maximal correlations in both bases together with Lorentz covariance for all four Bell states. On the other hand we know that the Wigner phase depends only on the direction of the momentum, so for parallel wave vectors this would not effect the detection.

Observable (535) does not always give eigenvalue $+1$ for one photon fields polarized under α angle and -1 for fields polarized under angle α'. Taking under consideration this eigenvalue problem we see that

$$\begin{aligned} Y_\alpha(N)|\Psi_\alpha(N,1)\rangle &= \int_\Omega d\Gamma(\boldsymbol{l})\ Y_\alpha(\boldsymbol{l}, N) \int d\Gamma(\boldsymbol{k})\ \Psi(\boldsymbol{k}, n_\alpha) a^\dagger_\alpha(\boldsymbol{k}, N)|O(N)\rangle \\ &= \frac{1}{\sqrt{2}}\int_\Omega d\Gamma(\boldsymbol{l}) \int d\Gamma(\boldsymbol{k})\ \Psi(\boldsymbol{k}, n_\alpha) \sum_{n=1}^{N}\left[\left(\sum_{s=\pm} e^{2is\alpha}|\boldsymbol{l}\rangle\langle\boldsymbol{l}| \otimes a^\dagger_{-s} a_s\right), \left(\sum_{s'=\pm} |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes a^\dagger_{s'} e^{-is'\alpha(\boldsymbol{k})}\right)\right]^{(n)} |O(N)\rangle \\ &= \frac{1}{\sqrt{2}}\int_\Omega d\Gamma(\boldsymbol{l}) \int d\Gamma(\boldsymbol{k})\ \delta_\Gamma(\boldsymbol{k}, \boldsymbol{l})\Psi(\boldsymbol{k}, n_\alpha) \sum_{n=1}^{N}\left(\sum_{s,s'=\pm} |\boldsymbol{l}\rangle\langle\boldsymbol{l}| \otimes a^\dagger_{-s}\delta_{ss'} e^{-is'\alpha(\boldsymbol{k})} e^{2is\alpha}\right)^{(n)} |O(N)\rangle \\ &= \frac{1}{\sqrt{2}}\int_\Omega d\Gamma(\boldsymbol{l})\ \Psi(\boldsymbol{l}, n_\alpha) \sum_{n=1}^{N}\left(\sum_{s=\pm} |\boldsymbol{l}\rangle\langle\boldsymbol{l}| \otimes a^\dagger_s e^{is\alpha(\boldsymbol{l})} e^{-2is\alpha}\right)^{(n)} |O(N)\rangle. \end{aligned} \tag{536}$$

So first we have to assume that support of the wave function $\Psi(\boldsymbol{l}, n_\alpha)$ is embedded in Ω and further within this angle the dependence on momentum of the polarization can be neglected, so $\alpha(\boldsymbol{l}) = \alpha$. Then the eigenvalue will be $+1$, i.e.

$$Y_\alpha(N)|\Psi_\alpha(N,1)\rangle \quad = \quad +1|\Psi_\alpha(N,1)\rangle. \tag{537}$$

For fields polarized under a perpendicular angle, in analogy to the previous calculus, we get

$$\begin{aligned}
Y_\alpha(N)|\Psi_{\alpha'}(N,1)\rangle \;=\; & \int_\Omega d\Gamma(\boldsymbol{l})\; Y_\alpha(\boldsymbol{l},N) \int d\Gamma(\boldsymbol{k})\; \Psi(\boldsymbol{k}, n_{\alpha'}) a^\dagger_{\alpha'}(\boldsymbol{k},N)|O(N)\rangle \\
= \; & \frac{1}{\sqrt{2}} \int_\Omega d\Gamma(\boldsymbol{l}) \; \int d\Gamma(\boldsymbol{k}) \; \Psi(\boldsymbol{k}, n_{\alpha'}) \\
\times \; & \sum_{n=1}^{N} \left[\left(\sum_{s=\pm} e^{2is\alpha} |\boldsymbol{l}\rangle\langle\boldsymbol{l}| \otimes a^\dagger_{-s} a_s \right), \left(\sum_{s'=\pm} |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes a^\dagger_{s'} e^{-is'\alpha'(\boldsymbol{k})} \right) \right]^{(n)} |O(N)\rangle \\
= \; & \frac{1}{\sqrt{2}} \int_\Omega d\Gamma(\boldsymbol{l}) \; \int d\Gamma(\boldsymbol{k}) \; \delta_\Gamma(\boldsymbol{k},\boldsymbol{l}) \Psi(\boldsymbol{k}, n_{\alpha'}) \sum_{n=1}^{N} \left(\sum_{s,s'=\pm} |\boldsymbol{l}\rangle\langle\boldsymbol{l}| \otimes a^\dagger_{-s} \delta_{ss'} e^{-is'\alpha'(\boldsymbol{k})} e^{2is\alpha} \right)^{(n)} |O(N)\rangle \\
= \; & -\frac{1}{\sqrt{2}} \int_\Omega d\Gamma(\boldsymbol{l}) \; \Psi(\boldsymbol{l}, n_{\alpha'}) \sum_{n=1}^{N} \left(\sum_{s=\pm} |\boldsymbol{l}\rangle\langle\boldsymbol{l}| \otimes a^\dagger_{s} e^{is\alpha'(\boldsymbol{l})} e^{-2is\alpha'} \right)^{(n)} |O(N)\rangle.
\end{aligned} \tag{538}$$

Again, assuming the wave function is all concentrated within Ω and within this angle spread the dependence on momentum of the polarization angle can be neglected, we get

$$Y_\alpha(N)|\Psi_{\alpha'}(N,1)\rangle \quad = \quad -1|\Psi_{\alpha'}(N,1)\rangle. \tag{539}$$

8.2 Correlation function for two-photon states

Now let us consider two observers in the same inertial frame. Alice measures α oriented photons and Bob β oriented ones. Their observables are $Y_\alpha(N)$ and $Y_\beta(N)$ respectively. In a more realistic case the localization of the photon detectors leads to a momentum solid angle spread $\boldsymbol{l} \in \Omega,\ \boldsymbol{l}' \in \Omega'$ respectively. Then the normalized correlation function for an arbitrary two state photon will be then given by:

$$\frac{\langle O(N)|\Psi(N)^\dagger Y_\beta(N) Y_\alpha(N) \Psi(N)|O(N)\rangle}{\langle O(N)|\Psi(N)^\dagger \Psi(N)|O(N)\rangle}. \tag{540}$$

First we will consider a two-photon field operator $\Psi(N)$ (444). The commutation relations for the yes-no observable and the two-photon state are derived explicitly in appendix (I.6) and (I.7):

$$[Y_\alpha(\boldsymbol{l},N), \Psi(N)] \quad = \quad 2 \sum_{s,s'=\pm} e^{2is\alpha} \int d\Gamma(\boldsymbol{k}) \; \psi_{ss'}(\boldsymbol{l},\boldsymbol{k}) a_{-s}(\boldsymbol{l},N)^\dagger a_{s'}(\boldsymbol{k},N)^\dagger, \tag{541}$$

$$[Y_\alpha(\boldsymbol{l},N), \Psi(N)^\dagger] \quad = \quad -2 \sum_{s,s'=\pm} e^{-2is\alpha} \int d\Gamma(\boldsymbol{k}) \; \bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{k}) a_{-s}(\boldsymbol{l},N) a_{s'}(\boldsymbol{k},N). \tag{542}$$

Using these formulas we get an unnormalized ERP average of the form

$$\begin{aligned}
& \langle O(N)|\Psi(N)^\dagger Y_\beta(N) Y_\alpha(N) \Psi(N)|O(N)\rangle \\
= \; & 4 \sum_{ss'=\pm} \int_\Omega d\Gamma(\boldsymbol{l}) \int_{\Omega'} d\Gamma(\boldsymbol{l}') \; e^{-2i(s\beta+s'\alpha)} \bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{l}') \psi_{-s'-s}(\boldsymbol{l}',\boldsymbol{l}) \langle O(N)|I(\boldsymbol{l},N) I(\boldsymbol{l}',N)|O(N)\rangle \\
+ \; & 4 \sum_{ss'=\pm} \int_\Omega d\Gamma(\boldsymbol{l}) \int_{\Omega'} d\Gamma(\boldsymbol{l}') \int d\Gamma(\boldsymbol{k}) \; e^{-2i(s\beta-s\alpha)} \bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{k}) \psi_{ss'}(\boldsymbol{l}',\boldsymbol{k}) \delta_\Gamma(\boldsymbol{l},\boldsymbol{l}') \langle O(N)|I(\boldsymbol{k},N) I(\boldsymbol{l},N)|O(N)\rangle.
\end{aligned} \tag{543}$$

This formula is derived in detail in appendix (J.3). In the case of disjoint detectors, i.e. $\Omega \cap \Omega' = \emptyset$, just one part of (543) has contribution to the EPR average, so that

$$\begin{aligned}
&\langle O(N)|\Psi(N)^\dagger Y_\beta(N) Y_\alpha(N) \Psi(N)|O(N)\rangle \\
= \;& 4 \sum_{ss'=\pm} \int_{\Omega} d\Gamma(\boldsymbol{l}) \int_{\Omega'} d\Gamma(\boldsymbol{l}')\; e^{-2i(s\beta+s'\alpha)} \bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{l}') \psi_{-s'-s}(\boldsymbol{l}',\boldsymbol{l}) \langle O(N)|I(\boldsymbol{l},N) I(\boldsymbol{l}',N)|O(N)\rangle \\
= \;& \frac{4(N-1)}{N} \sum_{ss'=\pm} \int_{\Omega} d\Gamma(\boldsymbol{l}) \int_{\Omega'} d\Gamma(\boldsymbol{l}')\; e^{-2i(s\beta+s'\alpha)} \bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{l}') \psi_{-s'-s}(\boldsymbol{l}',\boldsymbol{l}) Z(\boldsymbol{l}) Z(\boldsymbol{l}'). \qquad (544)
\end{aligned}$$

Now let us use the symmetry condition (447) and take a closer look at part:

$$\begin{aligned}
&\sum_{ss'=\pm} e^{-2i(s\beta+s'\alpha)} \bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{l}') \psi_{-s-s'}(\boldsymbol{l},\boldsymbol{l}') \;=\; \sum_{ss'=\pm} e^{2i(s\beta+s'\alpha)} \bar{\psi}_{-s-s'}(\boldsymbol{l},\boldsymbol{l}') \psi_{ss'}(\boldsymbol{l},\boldsymbol{l}') \\
= \;& 2\cos 2(\beta+\alpha) \Re\left(\bar{\psi}_{++}(\boldsymbol{l},\boldsymbol{l}') \psi_{--}(\boldsymbol{l},\boldsymbol{l}')\right) + 2\sin 2(\beta+\alpha) \Im\left(\bar{\psi}_{++}(\boldsymbol{l},\boldsymbol{l}') \psi_{--}(\boldsymbol{l},\boldsymbol{l}')\right) \\
+ \;& 2\cos 2(\beta-\alpha) \Re\left(\bar{\psi}_{+-}(\boldsymbol{l},\boldsymbol{l}') \psi_{-+}(\boldsymbol{l},\boldsymbol{l}')\right) + 2\sin 2(\beta-\alpha) \Im\left(\bar{\psi}_{+-}(\boldsymbol{l},\boldsymbol{l}') \psi_{-+}(\boldsymbol{l},\boldsymbol{l}')\right). \qquad (545)
\end{aligned}$$

This part is purely real and after some basic manipulations shown in appendix (K.1) we come to the form (545). This calculus was done mostly to bring up the $\cos 2(\beta \pm \alpha)$ part, known from literature. So the unnormalized EPR average for a two-photon state can be written in the form

$$\begin{aligned}
&\langle O(N)|\Psi(N)^\dagger Y_\beta(N) Y_\alpha(N) \Psi(N)|O(N)\rangle \\
= \;& \frac{8(N-1)}{N} \cos 2(\beta+\alpha) \int_{\Omega} d\Gamma(\boldsymbol{l}) \int_{\Omega'} d\Gamma(\boldsymbol{l}')\; \Re\left(\bar{\psi}_{++}(\boldsymbol{l},\boldsymbol{l}') \psi_{--}(\boldsymbol{l},\boldsymbol{l}')\right) Z(\boldsymbol{l}) Z(\boldsymbol{l}') \\
+ \;& \frac{8(N-1)}{N} \sin 2(\beta+\alpha) \int_{\Omega} d\Gamma(\boldsymbol{l}) \int_{\Omega'} d\Gamma(\boldsymbol{l}')\; \Im\left(\bar{\psi}_{++}(\boldsymbol{l},\boldsymbol{l}') \psi_{--}(\boldsymbol{l},\boldsymbol{l}')\right) Z(\boldsymbol{l}) Z(\boldsymbol{l}') \\
+ \;& \frac{8(N-1)}{N} \cos 2(\beta-\alpha) \int_{\Omega} d\Gamma(\boldsymbol{l}) \int_{\Omega'} d\Gamma(\boldsymbol{l}')\; \Re\left(\bar{\psi}_{+-}(\boldsymbol{l},\boldsymbol{l}') \psi_{-+}(\boldsymbol{l},\boldsymbol{l}')\right) Z(\boldsymbol{l}) Z(\boldsymbol{l}') \\
+ \;& \frac{8(N-1)}{N} \sin 2(\beta-\alpha) \int_{\Omega} d\Gamma(\boldsymbol{l}) \int_{\Omega'} d\Gamma(\boldsymbol{l}')\; \Im\left(\bar{\psi}_{+-}(\boldsymbol{l},\boldsymbol{l}') \psi_{-+}(\boldsymbol{l},\boldsymbol{l}')\right) Z(\boldsymbol{l}) Z(\boldsymbol{l}'). \qquad (546)
\end{aligned}$$

8.3 Correlation function for maximally anti-correlated in circular polarization basis states

Now let us consider the correlation function for Bell states, starting from maximally anti-correlated in circular polarization basis field operator $\Psi_1(N)$ (453). In a realistic case when the localization of the photon detector leads to a momentum solid angle spread $\boldsymbol{l} \in \Omega,\ \boldsymbol{l}' \in \Omega'$ and for disjoint detectors $\Omega \cap \Omega' = \emptyset$: the correlation function reads:

$$\begin{aligned}
&\langle O(N)|\Psi_1(N)^\dagger Y_\beta(N) Y_\alpha(N) \Psi_1(N)|O(N)\rangle \\
= \;& \frac{8(N-1)}{N} \cos 2(\beta-\alpha) \int_{\Omega} d\Gamma(\boldsymbol{l}) \int_{\Omega'} d\Gamma(\boldsymbol{l}')\; \Re\left(\bar{\psi}_{+-}(\boldsymbol{l},\boldsymbol{l}') \psi_{-+}(\boldsymbol{l},\boldsymbol{l}')\right) Z(\boldsymbol{l}) Z(\boldsymbol{l}') \\
+ \;& \frac{8(N-1)}{N} \sin 2(\beta-\alpha) \int_{\Omega} d\Gamma(\boldsymbol{l}) \int_{\Omega'} d\Gamma(\boldsymbol{l}')\; \Im\left(\bar{\psi}_{+-}(\boldsymbol{l},\boldsymbol{l}') \psi_{-+}(\boldsymbol{l},\boldsymbol{l}')\right) Z(\boldsymbol{l}) Z(\boldsymbol{l}'). \qquad (547)
\end{aligned}$$

For the field operator $\Psi_{11}(N)$ we will use the condition on the field and polarization angle (455), and then the EPR average can be written in the form

$$\begin{aligned}
&\langle O(N)|\Psi_{11}(N)^\dagger Y_\beta(N)Y_\alpha(N)\Psi_{11}(N)|O(N)\rangle \\
= \quad & \frac{-8(N-1)}{N}\cos 2(\beta-\alpha)\int\limits_\Omega d\Gamma(\boldsymbol{l})\int\limits_{\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\theta_{11}(\boldsymbol{l})-\theta_{11}(\boldsymbol{l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
- \quad & \frac{8(N-1)}{N}\sin 2(\beta-\alpha)\int\limits_\Omega d\Gamma(\boldsymbol{l})\int\limits_{\Omega'} d\Gamma(\boldsymbol{l}')\ \sin 2(\theta_{11}(\boldsymbol{l})-\theta_{11}(\boldsymbol{l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
= \quad & \frac{-8(N-1)}{N}\int\limits_\Omega d\Gamma(\boldsymbol{l})\int\limits_{\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-\theta_{11}(\boldsymbol{l})+\theta_{11}(\boldsymbol{l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}'). \qquad (548)
\end{aligned}$$

Furthermore, the normalized EPR average reads

$$\begin{aligned}
&\frac{\langle O(N)|\Psi_{11}(N)^\dagger Y_\beta(N)Y_\alpha(N)\Psi_{11}(N)|O(N)\rangle}{\langle O(N)|\Psi_{11}(N)^\dagger\Psi_{11}(N)|O(N)\rangle} \\
= \quad & \frac{-2(N-1)\int_\Omega d\Gamma(\boldsymbol{l})\ \int_{\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-\theta_{11}(\boldsymbol{l})+\theta_{11}(\boldsymbol{l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}')}{\int d\Gamma(\boldsymbol{k})|\psi_{+-}(\boldsymbol{k},\boldsymbol{k})|^2 Z(\boldsymbol{k})+(N-1)\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|\psi_{+-}(\boldsymbol{k},\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}')}. \qquad (549)
\end{aligned}$$

Now let us use the explicit values of $\psi_{+-}(\boldsymbol{k},\boldsymbol{k}')$ (490), taken from the Lorentz covariance condition of such two-photon field

$$\begin{aligned}
&\frac{\langle O(N)|\Psi_{11}(N)^\dagger Y_\beta(N)Y_\alpha(N)\Psi_{11}(N)|O(N)\rangle}{\langle O(N)|\Psi_{11}(N)^\dagger\Psi_{11}(N)|O(N)\rangle} \\
= \quad & \frac{-2(N-1)\int_\Omega d\Gamma(\boldsymbol{l})\ \int_{\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-\theta_{11}(\boldsymbol{l})+\theta_{11}(\boldsymbol{l}'))|m_a(\boldsymbol{l})\bar{m}^a(\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}')}{1+(N-1)\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|m_a(\boldsymbol{k})\bar{m}^a(\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}')}. \qquad (550)
\end{aligned}$$

We can see that the term $\int d\Gamma(\boldsymbol{k})|\psi_{+-}(\boldsymbol{k},\boldsymbol{k})|^2 Z(\boldsymbol{k})=1$ and this makes the EPR average dependent on the N parameter. In the $N\to\infty$ limit it takes the form

$$\begin{aligned}
&\lim_{N\to\infty}\frac{\langle O(N)|\Psi_{11}(N)^\dagger Y_\beta(N)Y_\alpha(N)\Psi_{11}(N)|O(N)\rangle}{\langle O(N)|\Psi_{11}(N)^\dagger\Psi_{11}(N)|O(N)\rangle} \\
= \quad & \frac{-2\int_\Omega d\Gamma(\boldsymbol{l})\ \int_{\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-\theta_{11}(\boldsymbol{l})+\theta_{11}(\boldsymbol{l}'))|m_a(\boldsymbol{l})\bar{m}^a(\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}')}{\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|m_a(\boldsymbol{k})\bar{m}^a(\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}')}. \qquad (551)
\end{aligned}$$

For the field operator $\Psi_{12}(N)$ (460), we will use the (459) condition on the field and the polarization angle, so that

$$\begin{aligned}
&\langle O(N)|\Psi_{12}(N)^\dagger Y_\beta(N)Y_\alpha(N)\Psi_{12}(N)|O(N)\rangle \\
= \quad & \frac{8(N-1)}{N}\cos 2(\beta-\alpha)\int_\Omega d\Gamma(\boldsymbol{l})\int_{\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\theta_{12}(\boldsymbol{l})-\theta_{12}(\boldsymbol{l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
+ \quad & \frac{8(N-1)}{N}\sin 2(\beta-\alpha)\int_\Omega d\Gamma(\boldsymbol{l})\int_{\Omega'} d\Gamma(\boldsymbol{l}')\ \sin 2(\theta_{12}(\boldsymbol{l})-\theta_{12}(\boldsymbol{l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
= \quad & \frac{8(N-1)}{N}\int_\Omega d\Gamma(\boldsymbol{l})\int_{\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-\theta_{12}(\boldsymbol{l})+\theta_{12}(\boldsymbol{l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}'). \qquad (552)
\end{aligned}$$

Then the normalized EPR average for $\Psi_{12}(N)$ states reads

$$\begin{aligned}
&\frac{\langle O(N)|\Psi_{12}(N)^\dagger Y_\beta(N)Y_\alpha(N)\Psi_{12}(N)|O(N)\rangle}{\langle O(N)|\Psi_{12}(N)^\dagger\Psi_{12}(N)|O(N)\rangle} \\
= \quad & \frac{2(N-1)\int_\Omega d\Gamma(\boldsymbol{l})\ \int_{\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-\theta_{12}(\boldsymbol{l})+\theta_{12}(\boldsymbol{l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}')}{\int d\Gamma(\boldsymbol{k})|\psi_{+-}(\boldsymbol{k},\boldsymbol{k})|^2 Z(\boldsymbol{k})+(N-1)\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|\psi_{+-}(\boldsymbol{k},\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}')}, \qquad (553)
\end{aligned}$$

and similarly for the explicit values of $\psi_{+-}(\boldsymbol{k},\boldsymbol{k}')$ (504) we get

$$\begin{aligned}
&\frac{\langle O(N)|\Psi_{12}(N)^\dagger Y_\beta(N)Y_\alpha(N)\Psi_{12}(N)|O(N)\rangle}{\langle O(N)|\Psi_{12}(N)^\dagger\Psi_{12}(N)|O(N)\rangle} \\
= \quad & \frac{2(N-1)\int_\Omega d\Gamma(\boldsymbol{l})\ \int_{\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-\theta_{12}(\boldsymbol{l})+\theta_{12}(\boldsymbol{l}'))|m_a(\boldsymbol{l})\bar{m}^a(\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}')}{1+(N-1)\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|m_a(\boldsymbol{k})\bar{m}^a(\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}')}. \qquad (554)
\end{aligned}$$

Also the EPR average for $\Psi_{12}(N)$ becomes dependent on the N parameter and in the $N \to \infty$ limit becomes

$$\begin{aligned} & \lim_{N\to\infty} \frac{\langle O(N)|\Psi_{12}(N)^\dagger Y_\beta(N) Y_\alpha(N) \Psi_{12}(N)|O(N)\rangle}{\langle O(N)|\Psi_{12}(N)^\dagger \Psi_{12}(N)|O(N)\rangle} \\ = & \frac{2\int_\Omega d\Gamma(\boldsymbol{l}) \ \int_{\Omega'} d\Gamma(\boldsymbol{l}') \ \cos 2(\beta - \alpha - \theta_{12}(\boldsymbol{l}) + \theta_{12}(\boldsymbol{l}'))|m_a(\boldsymbol{l})\bar{m}^a(\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}')}{\int d\Gamma(\boldsymbol{k}) d\Gamma(\boldsymbol{k}')|m_a(\boldsymbol{k})\bar{m}^a(\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}')}. \end{aligned} \tag{555}$$

Furthermore, using relation on the polarization angles $\theta_{11}(\boldsymbol{k})$ and $\theta_{12}(\boldsymbol{k})$ (461) we see that

$$\frac{\langle O(N)|\Psi_{11}(N)^\dagger Y_\beta(N) Y_\alpha(N) \Psi_{11}(N)|O(N)\rangle}{\langle O(N)|\Psi_{11}(N)^\dagger \Psi_{11}(N)|O(N)\rangle} = \frac{\langle O(N)|\Psi_{12}(N)^\dagger Y_\beta(N) Y_\alpha(N) \Psi_{12}(N)|O(N)\rangle}{\langle O(N)|\Psi_{12}(N)^\dagger \Psi_{12}(N)|O(N)\rangle}. \tag{556}$$

8.4 Correlation function for maximally correlated in circular polarization basis states

Now we will follow the same calculations as from the previous section, only this time for the field operators corresponding to states maximally correlated in circular basis. For such operators let us consider $\Psi_2(N)$ (464). Again we are assuming a realistic case when the localization of the photon detector leads to a momentum solid angle spread $\boldsymbol{l} \in \Omega$, $\boldsymbol{l}' \in \Omega'$ and disjoint detectors $\Omega \cap \Omega' = \emptyset$, so the correlation function reads:

$$\begin{aligned} & \langle O(N)|\Psi_2(N)^\dagger Y_\beta(N) Y_\alpha(N) \Psi_2(N)|O(N)\rangle \\ = & \frac{8(N-1)}{N} \cos 2(\beta+\alpha) \int\limits_\Omega d\Gamma(\boldsymbol{l}) \int\limits_{\Omega'} d\Gamma(\boldsymbol{l}') \ \Re\left(\bar{\psi}_{++}(\boldsymbol{l},\boldsymbol{l}')\psi_{--}(\boldsymbol{l},\boldsymbol{l}')\right) Z(\boldsymbol{l})Z(\boldsymbol{l}') \\ + & \frac{8(N-1)}{N} \sin 2(\beta+\alpha) \int\limits_\Omega d\Gamma(\boldsymbol{l}) \int\limits_{\Omega'} d\Gamma(\boldsymbol{l}') \ \Im\left(\bar{\psi}_{++}(\boldsymbol{l},\boldsymbol{l}')\psi_{--}(\boldsymbol{l},\boldsymbol{l}')\right) Z(\boldsymbol{l})Z(\boldsymbol{l}'). \end{aligned} \tag{557}$$

For the field operator $\Psi_{21}(N)$ (467), we will use the (466) condition on the field and polarization angle

$$\begin{aligned} & \langle O(N)|\Psi_{21}(N)^\dagger Y_\beta(N) Y_\alpha(N) \Psi_{21}(N)|O(N)\rangle \\ = & \frac{-8(N-1)}{N} \cos 2(\beta+\alpha) \int\limits_\Omega d\Gamma(\boldsymbol{l}) \int\limits_{\Omega'} d\Gamma(\boldsymbol{l}') \ \cos 2(\theta(\boldsymbol{l})+\theta(\boldsymbol{l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\ + & \frac{-8(N-1)}{N} \sin 2(\beta+\alpha) \int\limits_\Omega d\Gamma(\boldsymbol{l}) \int\limits_{\Omega'} d\Gamma(\boldsymbol{l}') \ \sin 2(\theta_{21}(\boldsymbol{l})+\theta_{21}(\boldsymbol{l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\ = & \frac{-8(N-1)}{N} \int\limits_\Omega d\Gamma(\boldsymbol{l}) \int\limits_{\Omega'} d\Gamma(\boldsymbol{l}') \ \cos 2(\beta+\alpha-\theta_{21}(\boldsymbol{l})-\theta_{21}(\boldsymbol{l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}'). \end{aligned} \tag{558}$$

Then the normalized EPR average for $\Psi_{21}(N)$ reads

$$\begin{aligned} & \frac{\langle O(N)|\Psi_{21}(N)^\dagger Y_\beta(N) Y_\alpha(N) \Psi_{21}(N)|O(N)\rangle}{\langle O(N)|\Psi_{21}(N)^\dagger \Psi_{21}(N)|O(N)\rangle} \\ = & \frac{-2(N-1)\int_\Omega d\Gamma(\boldsymbol{l}) \ \int_{\Omega'} d\Gamma(\boldsymbol{l}') \ \cos 2(\beta+\alpha-\theta_{21}(\boldsymbol{l})-\theta_{21}(\boldsymbol{l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}')}{\int d\Gamma(\boldsymbol{k})|\psi_{--}(\boldsymbol{k},\boldsymbol{k})|^2 Z(\boldsymbol{k}) + (N-1)\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|\psi_{--}(\boldsymbol{k},\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}')}. \end{aligned} \tag{559}$$

Using the explicit value of $\psi_{--}(\boldsymbol{k},\boldsymbol{k}')$ and knowing that $\psi_{--}(\boldsymbol{k},\boldsymbol{k}) = 0$, we come to the EPR average that does not depend on the N parameter, i.e.

$$\begin{aligned} & \frac{\langle O(N)|\Psi_{21}(N)^\dagger Y_\beta(N) Y_\alpha(N) \Psi_{21}(N)|O(N)\rangle}{\langle O(N)|\Psi_{21}(N)^\dagger \Psi_{21}(N)|O(N)\rangle} \\ = & \frac{-2\int_\Omega d\Gamma(\boldsymbol{l}) \ \int_{\Omega'} d\Gamma(\boldsymbol{l}') \ \cos 2(\beta+\alpha-\theta_{21}(\boldsymbol{l})-\theta_{21}(\boldsymbol{l}'))|m_a(\boldsymbol{l})m^a(\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}')}{\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|m_a(\boldsymbol{k})m^a(\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}')}. \end{aligned} \tag{560}$$

For the field operator $\Psi_{22}(N)$ (471), we will use the condition on the field and the polarization angle (470), so that the unnormalized EPR average can be written in the form

$$\begin{aligned}
& \langle O(N)|\Psi_{22}(N)^\dagger Y_\beta(N) Y_\alpha(N)\Psi_{22}(N)|O(N)\rangle \\
= & \frac{8(N-1)}{N}\cos 2(\beta+\alpha)\int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Omega'} d\Gamma(\boldsymbol{l}') \ \cos 2(\theta_{22}(\boldsymbol{l})+\theta_{22}(\boldsymbol{l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
+ & \frac{8(N-1)}{N}\sin 2(\beta+\alpha)\int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Omega'} d\Gamma(\boldsymbol{l}') \ \sin 2(\theta_{22}(\boldsymbol{l})+\theta_{22}(\boldsymbol{l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
= & \frac{8(N-1)}{N}\int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Omega'} d\Gamma(\boldsymbol{l}') \ \cos 2(\beta+\alpha-\theta_{22}(\boldsymbol{l})-\theta_{22}(\boldsymbol{l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}'). \qquad (561)
\end{aligned}$$

Then the normalized EPR average for $\Psi_{22}(N)$ Bell state reads

$$\begin{aligned}
& \frac{\langle O(N)|\Psi_{22}(N)^\dagger Y_\beta(N) Y_\alpha(N)\Psi_{22}(N)|O(N)\rangle}{\langle O(N)|\Psi_{22}(N)^\dagger \Psi_{22}(N)|O(N)\rangle} \\
= & \frac{2(N-1)\int_\Omega d\Gamma(\boldsymbol{l}) \ \int_{\Omega'} d\Gamma(\boldsymbol{l}') \ \cos 2(\beta+\alpha-\theta_{22}(\boldsymbol{l})-\theta_{22}(\boldsymbol{l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}')}{\int d\Gamma(\boldsymbol{k})|\psi_{--}(\boldsymbol{k},\boldsymbol{k})|^2 Z(\boldsymbol{k}) + (N-1)\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|\psi_{--}(\boldsymbol{k},\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}')}. \qquad (562)
\end{aligned}$$

It should be stressed that also for the $\Psi_{22}(N)$ field operator the EPR average does not depend on the N parameter

$$\begin{aligned}
& \frac{\langle O(N)|\Psi_{22}(N)^\dagger Y_\beta(N) Y_\alpha(N)\Psi_{22}(N)|O(N)\rangle}{\langle O(N)|\Psi_{22}(N)^\dagger \Psi_{22}(N)|O(N)\rangle} \\
= & \frac{2\int_\Omega d\Gamma(\boldsymbol{l}) \ \int_{\Omega'} d\Gamma(\boldsymbol{l}') \ \cos 2(\beta+\alpha-\theta_{22}(\boldsymbol{l})-\theta_{22}(\boldsymbol{l}'))|m_a(\boldsymbol{l})m^a(\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}')}{\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|m_a(\boldsymbol{k})m^a(\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}')}. \qquad (563)
\end{aligned}$$

Finally, using the relation between the polarization angles $\theta_{21}(\boldsymbol{k})$ and $\theta_{22}(\boldsymbol{k})$ (472), we see that

$$\frac{\langle O(N)|\Psi_{21}(N)^\dagger Y_\beta(N) Y_\alpha(N)\Psi_{21}(N)|O(N)\rangle}{\langle O(N)|\Psi_{21}(N)^\dagger \Psi_{21}(N)|O(N)\rangle} = \frac{\langle O(N)|\Psi_{22}(N)^\dagger Y_\beta(N) Y_\alpha(N)\Psi_{22}(N)|O(N)\rangle}{\langle O(N)|\Psi_{22}(N)^\dagger \Psi_{22}(N)|O(N)\rangle}. \qquad (564)$$

8.5 Results and conclusions

This chapter contains new results. The EPR correlation functions which describes measurements on detectors, for all four Bell states where calculated and from this two main conclusion can be made. First involves the N parameter. In reducible representations the N parameter does not necessary have to go to infinity, since each oscillator is a superposition of already infinitely many different momentum states. If we made an assumption that the N parameter is a finite large number, it would have had an influence on the outcome of the EPR average for states maximally anti-correlated in circular basis. Like shown in section 8.3 the EPR average for maximally anti-correlated in circular polarization basis states depends on the N parameter. The extra term in the denominator of the EPR averages for maximally anti-correlated in circular basis Bell states corresponding to the $\Psi_1(N)$ field operator may have influence on the outcome compared with the maximally correlated in circular basis Bell states corresponding to field operator $\Psi_2(N)$. Putting it another way, if any experiments confirmed a smaller outcome of the EPR average for maximally anti-correlated in circular basis states comparing with maximally correlated in circular basis states, it could have spoken in favor for the N parameter being a finite number.

The second conclusion involves the polarization angle which for this representation is dependent on momentum. For example let us take the EPR average for the correlated in circular polarization basis field operator $\Psi_2(N)$

$$\frac{2\int_\Omega d\Gamma(\boldsymbol{l}) \ \int_{\Omega'} d\Gamma(\boldsymbol{l}') \ \cos 2(\beta+\alpha-\theta_{22}(\boldsymbol{l})-\theta_{22}(\boldsymbol{l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}')}{\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|\psi_{--}(\boldsymbol{k},\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}')}. \qquad (565)$$

Number 2 in the numerator may look suspicious at first, but it can be shown that this comes from the symmetry of the $|\psi_{--}(\boldsymbol{k},\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}')$ term. Denoting $f(\boldsymbol{k},\boldsymbol{k}') = |\psi_{--}(\boldsymbol{k},\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}')$, we see that $f(\boldsymbol{k},\boldsymbol{k}') = f(\boldsymbol{k}',\boldsymbol{k})$, and

$$\begin{aligned}
& 2\int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta+\alpha-\theta_{22}(\boldsymbol{l})-\theta_{22}(\boldsymbol{l}'))f(\boldsymbol{l},\boldsymbol{l}') \\
= & \int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta+\alpha-\theta_{22}(\boldsymbol{l})-\theta_{22}(\boldsymbol{l}'))f(\boldsymbol{l},\boldsymbol{l}') \\
+ & \int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta+\alpha-\theta_{22}(\boldsymbol{l}')-\theta_{22}(\boldsymbol{l}))f(\boldsymbol{l}',\boldsymbol{l}) \\
= & \int_{(\Omega\times\Omega')\cup(\Omega'\times\Omega)} d\Gamma(\boldsymbol{l})\ d\Gamma(\boldsymbol{l}')\ \cos 2(\beta+\alpha-\theta_{22}(\boldsymbol{l})-\theta_{22}(\boldsymbol{l}'))f(\boldsymbol{l},\boldsymbol{l}').
\end{aligned} \tag{566}$$

We find that $(\Omega\times\Omega')\cup(\Omega'\times\Omega)\subset R^3\times R^3$, $f(\boldsymbol{k},\boldsymbol{k}')$ is always nonnegative and the cosine term is bounded, i.e. $|\cos 2(\beta+\alpha-\theta_{22}(\boldsymbol{l})-\theta_{22}(\boldsymbol{l}'))|\leqslant 1$, which implies that

$$\frac{\int_{(\Omega\times\Omega')\cup(\Omega'\times\Omega)} d\Gamma(\boldsymbol{l})\ d\Gamma(\boldsymbol{l}')\ \cos 2(\beta+\alpha-\theta_{22}(\boldsymbol{l})-\theta_{22}(\boldsymbol{l}'))f(\boldsymbol{l},\boldsymbol{l}')}{\int_{R^3\times R^3} d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')f(\boldsymbol{k},\boldsymbol{k}')} \leqslant 1. \tag{567}$$

As we can see the EPR average in such reducible representations with the polarization angle dependent on momentum may serve a shift of phase comparing with "standard theory models", other than that is hard to distinguish from "standard models" for $Z(\boldsymbol{k})$ being flat in the detectors' momentum solid angle spread.

9 EPR-type experiment under Lorentz transformation

In a situation where the two particles are co-moving, Bell's inequality has been discussed by Czachor [53] in 1997. In this paper mainly two aspects of EPR experiment in relativistic frame work were discussed: a possibility of using the experiment as an implicit test of a relativistic concept of centre-of-mas and the influence of relativistic effect on the degree of violation of Bells inequality. The conclusion was that the relativistic effects are relevant to the experiment where the degree of violation on Bells inequalities depends on the velocity of entangled particles.

Also in 1997 Suarez and Scardini [54], [55] pointed out that since the simultaneity of two events depends on the reference frame, the correlations between entangled photons may be affected by the motion of detectors.

The main difficulty for Lorentz transformation law of Bell states is the dependence of Wigner rotations $\Theta(\Lambda, \boldsymbol{k})$ on momentum. This was discussed by Peres, Scudo and Terno in paper [57] from 2002. They concluded that the spin density matrix for a single spin 1/2 particle is not invariant. When observed form a moving frame, Wigner rotations entangle the spin with the particles momentum distribution. In relativistic context spin and momentum are not independent degrees of freedom. It is not possible to change the inertial reference frame without changing the quantization axis of spin. In consequence Lorentz boosts introduce a transfer of entanglement between degrees on freedom. This could be useful for entanglement manipulation. Lorentz boosts act as a global transformation on spin and momentum not as a local transformation strictly on spin or strictly on momentum. In other words, the entanglement between spin and momentum alone may not be invariant, the entanglement of the entire field (spin and momentum) in invariant. The conclusion was that spin state for a particle is meaningless if it is not specified completely including momentum dependence.

Genrich and Adami [65] have shown that entanglement between the spins of a pair of spin-1/2 particles is carried over to the entanglement between the momenta of the particles by the Wigner rotation, even though the entanglement of the entire system is Lorentz invariant. It depends on the reference frame depending on the field of the pair. They also gave an example of a pair fully spin entangled in the rest frame but with a reduction of spin entanglement in other frames. Similarly they showed that there are pairs with spin entanglement increment when boosted.

In 2003 Bergou, Gingrich and Adami [66] calculated the entanglement between a pair of polarization entangled photons as a function of the reference frame. They showed that the transformation law for helicity-momentum eigenstates, produces a helicity-momentum phase. This phase decreases or increases entanglement of the pair depending on the boost direction, the rapidity and the spread of the beam.

Furthermore, Alsing and Milburn [63], [64] have argued that entanglement of a two-particle state is preserved under Lorentz transformations.

Also Ahn Lee and Hwang [67] studied Lorentz transformation of massive two-particle entangled quantum states. They concluded that to an observer in a moving frame the Bell states appear as rotations or linear combinations of Bell states in that frame.

Terashima and Ueda in papers [60], [61] considered a similar situation but discussed the EPR correlation rather than the entanglement using the spin-singlet state in terms of the state vector with factorable pure momentum eigenstates. They analyzed a situation in which measurements are performed by moving observers for mass and massless particles. They concluded that the entanglement is independent of the basis for the measurement, but the correlation depends on it. They pointed out that under certain conditions the perfect anti-correlation of an EPR pair of spins in the same direction is deteriorated in the moving observers frame due to the Wigner rotation, and have shown that the degree of the violation of Bell's inequality at first sight decreases with increasing the velocity of the observers if the directions of the measurement are fixed. However, this does not imply a breakdown of non-local correlation since the perfect anti-correlation is maintained in appropriately chosen different directions.

You, Wang, Yang, Niu, Ma and Xu in 2004 [74] discussed a relativistic version of Greenberger-Horn-Zeilinger (GHZ) experiment of massive particles. GHZ correlations provided for Bohm's version of EPR and are no longer statistical in principle. In their gedankenexperiment of relativistic GHZ they concluded that spin variables averages that are maximally correlated in the laboratory frame no longer appear so in the same direction seen from the moving frame however it is always possible to to find a different direction that shows perfect correlation of the GHZ state.

Friis, Bertlmann, Huber and Heinmayer [94] in 2010 considered spin and momentum degrees of freedom for spin-1/2 particles in a four-qubit system. They concluded that entanglement can not be generally considered to be Lorentz invariant and depends on the choice of reference frame and the partitions of considered four-qubits. They end their discussion with a question whether same conclusions can be made considering second quantization formalism.

Experimentally relativistic EPR-type multisimultaneity problem was tested by Stefanov, Zbinden, Gisin and Suarez [100] in 2002. From this experiment they concluded that no disappearance of correlations was observed.

In this chapter relativistic correlations of Bell states in the background of reducible representations of N-oscillator algebras will be considered. For relativistic considerations the field should be a Lorentz covariant one, and invariance of the four Bell states of the proposed model was shown in previous section 7. Let us consider two observers moving relative to each other with detectors that detect an EPR pair. As far as simultaneity of events is considered the moment of the collapse of the field will depend on the reference frame and observers may have a disagreement on whose detectors clicked first. Therefore for considerations made in this chapter the simultaneity problem is not an issue in EPR experiment, where in the experiment we ask a question about the correlations. In other words, although observers may not agree on whose detector clicked first, they will agree on the outcome of the experiment, i.e. the correlations. In this sense there in no preferred frame of reference.

This chapter is organized as follows. In section 9.1 the transformation rule for the detectors modeled by a yes-no observable is shown and a relativistic correlation function is derived for a two-photon state in the case where two detectors are transformed under Lorentz transformation in such a way that they still maintain in the same reference frame. These results will be used for evaluating EPR averages for the four Bell states in next sections. Further in 9.2 and 9.3 the same calculation are done for maximally anti-correlated and correlated states in circular basis respectively. In section 9.4 a relativistic correlation function is derived for a two-photon state in the case where just one of the detectors is transformed under Lorentz transformation. Finally in section 9.5 and in section 9.6 the same calculation are done for maximally anti-correlated and correlated states in circular basis respectively.

9.1 Relativistic correlation of a two-photon states - case 1

Within the framework of relativistic quantum field theory, let us consider the Einstein-Podolsky-Rosen (EPR) gedankenexperiment in which measurements on detectors are performed by moving observers. In this section we perform a Lorenz transformation on both detectors Alice's and Bob's, so that Alice and Bob are in the same reference frame, not moving with respect to each other. We will start with a two-photon state.

First let us notice that in detectors modeled by a yes-no observable (533) after a Lorentz transformation the α orientation angle is observed to be shifted by the Wigner phase $2\Theta(\Lambda,\boldsymbol{r})$.

$$\begin{aligned} U(\Lambda,0,N)^{\dagger}Y_{\alpha}(\boldsymbol{l},N)U(\Lambda,0,N) &= \sum_{n=1}^{N}\sum_{s=\pm}\Big(e^{2is\alpha}e^{-4is\Theta(\Lambda,\boldsymbol{l})}|\boldsymbol{\Lambda^{-1}l}\rangle\langle\boldsymbol{\Lambda^{-1}l}|\otimes a_{-s}^{\dagger}a_{s}\Big)^{(n)} \\ &= Y_{\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})}(\boldsymbol{\Lambda^{-1}l},N). \end{aligned} \tag{568}$$

This formula is derived explicitly in appendix (H.13). Also the commutation relations of the yes-no observable with a two-photon field operator (444) are derived in appendix (I.8) and (I.9):

$$\begin{aligned} [U(\Lambda,0,N)^{\dagger}Y_{\alpha}(\boldsymbol{l},N)U(\Lambda,0,N),\Psi(N)] &= [Y_{\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})}(\boldsymbol{\Lambda^{-1}l},N),\Psi(N)] \\ = 2\sum_{s,s'=\pm}e^{2is\alpha}e^{-4is\Theta(\Lambda,\boldsymbol{l})}\int d\Gamma(\boldsymbol{k})\ \psi_{ss'}(\boldsymbol{\Lambda^{-1}l},\boldsymbol{k})a_{-s}(\boldsymbol{\Lambda^{-1}l},N)^{\dagger}a_{s'}(\boldsymbol{k},N)^{\dagger}, \end{aligned} \tag{569}$$

$$
\begin{aligned}
&[U(\Lambda,0,N)^\dagger Y_\alpha(\boldsymbol{l},N)U(\Lambda,0,N),\Psi(N)^\dagger] \;=\; [Y_{\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})}(\boldsymbol{\Lambda^{-1}l},N),\Psi(N)^\dagger] \\
&= \; -2\sum_{s,s'=\pm} e^{-2is\alpha}e^{4is\Theta(\Lambda,\boldsymbol{l})}\int d\Gamma(\boldsymbol{k})\;\bar\psi_{ss'}(\boldsymbol{\Lambda^{-1}l},\boldsymbol{k})a_{-s}(\boldsymbol{\Lambda^{-1}l},N)a_{s'}(\boldsymbol{k},N). \qquad (570)
\end{aligned}
$$

Furthermore, under Lorentz transformation performed on both detectors we have an unnormalized EPR average of the form

$$
\begin{aligned}
&\langle O(N)|\Psi(N)^\dagger U(\Lambda,0,N)^\dagger Y_\beta(N)Y_\alpha(N)U(\Lambda,0,N)\Psi(N)|O(N)\rangle \\
=\;& 4\sum_{ss'=\pm}\int_\Omega d\Gamma(\boldsymbol{l})\int_{\Omega'} d\Gamma(\boldsymbol{l}')\; e^{-2i(s\beta+s'\alpha)}e^{4i(s\Theta(\Lambda,\boldsymbol{l})+s'\Theta(\Lambda,\boldsymbol{l}'))}\bar\psi_{ss'}(\boldsymbol{\Lambda^{-1}l},\boldsymbol{\Lambda^{-1}l}')\psi_{-s'-s}(\boldsymbol{\Lambda^{-1}l}',\boldsymbol{\Lambda^{-1}l}) \\
\times\;& \langle O(N)|I(\boldsymbol{\Lambda^{-1}l},N)I(\boldsymbol{\Lambda^{-1}l}',N)|O(N)\rangle \\
+\;& 4\sum_{ss'=\pm}\int_\Omega d\Gamma(\boldsymbol{l})\int_{\Omega'} d\Gamma(\boldsymbol{l}')\int d\Gamma(\boldsymbol{k})\; e^{-2i(s\beta-s\alpha)}e^{4i(s\Theta(\Lambda,\boldsymbol{l})-s\Theta(\Lambda,\boldsymbol{l}'))} \\
\times\;& \bar\psi_{ss'}(\boldsymbol{\Lambda^{-1}l},\boldsymbol{k})\psi_{ss'}(\boldsymbol{\Lambda^{-1}l}',\boldsymbol{k})\delta_\Gamma(\boldsymbol{\Lambda^{-1}l},\boldsymbol{\Lambda^{-1}l}')\langle O(N)|I(\boldsymbol{k},N)I(\boldsymbol{\Lambda^{-1}l},N)|O(N)\rangle. \qquad (571)
\end{aligned}
$$

This formula is also derived step by step in appendix (J.4). In the case of disjoint detectors just one part has contribution. Also from the transformation rule on the fields (482) we get:

$$
\begin{aligned}
&\langle O(N)|\Psi(N)^\dagger U(\Lambda,0,N)^\dagger Y_\beta(N)Y_\alpha(N)U(\Lambda,0,N)\Psi(N)|O(N)\rangle \\
=\;& 4\sum_{ss'=\pm}\int_\Omega d\Gamma(\boldsymbol{l})\int_{\Omega'} d\Gamma(\boldsymbol{l}')\; e^{-2i(s\beta+s'\alpha)}\bar\psi_{ss'}(\boldsymbol{l},\boldsymbol{l}')\psi_{-s-s'}(\boldsymbol{l},\boldsymbol{l}')\times\langle O(N)|I(\boldsymbol{\Lambda^{-1}l},N)I(\boldsymbol{\Lambda^{-1}l}',N)|O(N)\rangle \\
=\;& 4\sum_{ss'=\pm}\int_\Omega d\Gamma(\boldsymbol{l})\int_{\Omega'} d\Gamma(\boldsymbol{l}')\; e^{-2i(s\beta+s'\alpha)}\bar\psi_{ss'}(\boldsymbol{l},\boldsymbol{l}')\psi_{-s-s}(\boldsymbol{l},\boldsymbol{l}') \\
\times\;& \left(\frac{1}{N}\left(\bar O(\boldsymbol{\Lambda^{-1}l})O(\boldsymbol{\Lambda^{-1}l}')\delta_\Gamma(\boldsymbol{\Lambda^{-1}l},\boldsymbol{\Lambda^{-1}l}')+(N-1)Z(\boldsymbol{\Lambda^{-1}l})Z(\boldsymbol{\Lambda^{-1}l}')\right)\right) \\
=\;& \frac{4(N-1)}{N}\sum_{ss'=\pm}\int_\Omega d\Gamma(\boldsymbol{l})\int_{\Omega'} d\Gamma(\boldsymbol{l}')\; e^{-2i(s\beta+s'\alpha)}\bar\psi_{ss'}(\boldsymbol{l},\boldsymbol{l}')\psi_{-s-s'}(\boldsymbol{l},\boldsymbol{l}')Z(\boldsymbol{\Lambda^{-1}l})Z(\boldsymbol{\Lambda^{-1}l}') \\
=\;& \frac{4(N-1)}{N}\sum_{ss'=\pm}\int_{\Lambda\Omega} d\Gamma(\boldsymbol{l})\int_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\; e^{-2i(s\beta+s'\alpha)}\bar\psi_{ss'}(\boldsymbol{\Lambda l},\boldsymbol{\Lambda l}')\psi_{-s-s'}(\boldsymbol{\Lambda l},\boldsymbol{\Lambda l}')Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
=\;& \frac{4(N-1)}{N}\sum_{ss'=\pm}\int_{\Lambda\Omega} d\Gamma(\boldsymbol{l})\int_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\; e^{-2i(s\beta+s'\alpha)}e^{4i(s\Theta(\Lambda,\boldsymbol{\Lambda l})+s'\Theta(\Lambda,\boldsymbol{\Lambda l}'))}\bar\psi_{ss'}(\boldsymbol{l},\boldsymbol{l}')\psi_{-s-s'}(\boldsymbol{l},\boldsymbol{l}')Z(\boldsymbol{l})Z(\boldsymbol{l}').
\end{aligned}
$$

(572)

Now, just for bringing out the $\cos 2(\beta\pm\alpha)$ term known from literature, let us take a closer look at part:

$$
\begin{aligned}
&\sum_{ss'=\pm} e^{-2i(s\beta+s'\alpha)}e^{4i(s\Theta(\Lambda,\boldsymbol{\Lambda l})+s'\Theta(\Lambda,\boldsymbol{\Lambda l}'))}\bar\psi_{ss'}(\boldsymbol{l},\boldsymbol{l}')\psi_{-s-s'}(\boldsymbol{l},\boldsymbol{l}') \\
=\;& 2\cos 2(\beta+\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})-2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\Re\left(\bar\psi_{++}(\boldsymbol{l},\boldsymbol{l}')\psi_{--}(\boldsymbol{l},\boldsymbol{l}')\right) \\
+\;& 2\sin 2(\beta+\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})-2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\Im\left(\bar\psi_{++}(\boldsymbol{l},\boldsymbol{l}')\psi_{--}(\boldsymbol{l},\boldsymbol{l}')\right) \\
+\;& 2\cos 2(\beta-\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})+2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\Re\left(\bar\psi_{+-}(\boldsymbol{l},\boldsymbol{l}')\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')\right) \\
+\;& 2\sin 2(\beta-\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})+2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\Im\left(\bar\psi_{+-}(\boldsymbol{l},\boldsymbol{l}')\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')\right). \qquad (573)
\end{aligned}
$$

So the unnormalized EPR average, in the case when both detectors undergo the same Lorentz transformation, for a two-photon state can be written in the form

$$
\begin{aligned}
&\langle O(N)|\Psi(N)^{\dagger}U(\Lambda,0,N)^{\dagger}Y_{\beta}(N)Y_{\alpha}(N)U(\Lambda,0,N)\Psi(N)|O(N)\rangle \\
= &\ \frac{8(N-1)}{N}\int\limits_{\Lambda\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta+\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})-2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\Re\left(\bar{\psi}_{++}(\boldsymbol{l},\boldsymbol{l}')\psi_{--}(\boldsymbol{l},\boldsymbol{l}')\right)Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
+ &\ \frac{8(N-1)}{N}\int\limits_{\Lambda\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \sin 2(\beta+\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})-2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\Im\left(\bar{\psi}_{++}(\boldsymbol{l},\boldsymbol{l}')\psi_{--}(\boldsymbol{l},\boldsymbol{l}')\right)Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
+ &\ \frac{8(N-1)}{N}\int\limits_{\Lambda\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})+2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\Re\left(\bar{\psi}_{+-}(\boldsymbol{l},\boldsymbol{l}')\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')\right)Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
+ &\ \frac{8(N-1)}{N}\int\limits_{\Lambda\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \sin 2(\beta-\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})+2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\Im\left(\bar{\psi}_{+-}(\boldsymbol{l},\boldsymbol{l}')\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')\right)Z(\boldsymbol{l})Z(\boldsymbol{l}').
\end{aligned}
\tag{574}
$$

9.2 Relativistic correlation function for maximally anti-correlated in circular polarization basis states - case 1

Now let us consider such relativistic correlation function for the Bell states. We will start from the maximally anti-correlated states in circular polarization basis corresponding to the field operator $\Psi_1(N)$ (453). In a realistic case, when the localization of the photon detector leads to a momentum solid angle spread $\boldsymbol{l}\in\Omega,\ \boldsymbol{l}'\in\Omega'$ and the detectors are disjoint $\Omega\cap\Omega'=\emptyset$, the correlation function reads:

$$
\begin{aligned}
&\langle O(N)|\Psi_1(N)^{\dagger}U(\Lambda,0,N)^{\dagger}Y_{\beta}(N)Y_{\alpha}(N)U(\Lambda,0,N)\Psi_1(N)|O(N)\rangle \\
= &\ \frac{8(N-1)}{N}\int\limits_{\Lambda\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})+2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\Re\left(\bar{\psi}_{+-}(\boldsymbol{l},\boldsymbol{l}')\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')\right)Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
+ &\ \frac{8(N-1)}{N}\int\limits_{\Lambda\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \sin 2(\beta-\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})+2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\Im\left(\bar{\psi}_{+-}(\boldsymbol{l},\boldsymbol{l}')\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')\right)Z(\boldsymbol{l})Z(\boldsymbol{l}').
\end{aligned}
\tag{575}
$$

For the field operator $\Psi_{11}(N)$ (456), using the condition on the field and the polarization angle (455), we get

$$
\begin{aligned}
&\langle O(N)|\Psi_{11}(N)^{\dagger}U(\Lambda,0,N)^{\dagger}Y_{\beta}(N)Y_{\alpha}(N)U(\Lambda,0,N)\Psi_{11}(N)|O(N)\rangle \\
= &\ \frac{-8(N-1)}{N}\int\limits_{\Lambda\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})+2\Theta(\Lambda,\boldsymbol{\Lambda l}')) \\
\times &\ \cos 2(\theta_{11}(\boldsymbol{l})-\theta_{11}(\boldsymbol{l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
+ &\ \frac{-8(N-1)}{N}\int\limits_{\Lambda\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \sin 2(\beta-\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})+2\Theta(\Lambda,\boldsymbol{\Lambda l}')) \\
\times &\ \sin 2(\theta_{11}(\boldsymbol{l})-\theta_{11}(\boldsymbol{l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
= &\ \frac{-8(N-1)}{N}\int\limits_{\Lambda\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})+2\Theta(\Lambda,\boldsymbol{\Lambda l}')-\theta_{11}(\boldsymbol{l})+\theta_{11}(\boldsymbol{l}')) \\
\times &\ |\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
= &\ \frac{-8(N-1)}{N}\int\limits_{\Lambda\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-\theta_{11}(\boldsymbol{\Lambda l})+\theta_{11}(\boldsymbol{\Lambda l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \qquad (576) \\
= &\ \frac{-8(N-1)}{N}\int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-\theta_{11}(\boldsymbol{l})+\theta_{11}(\boldsymbol{l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{\Lambda^{-1}l})Z(\boldsymbol{\Lambda^{-1}l}').
\end{aligned}
\tag{577}
$$

One may look at this formula from two points of view: as a transformation on the detectors angle spread and the polarization angle $\theta_{11}(\boldsymbol{k})$ (576) or a transformation on the vacuum probability density (577). Then the normalized relativistic EPR average for $\Psi_{11}(N)$ reads

$$\begin{aligned}
&\frac{\langle O(N)|\Psi_{11}(N)^\dagger U(\Lambda,0,N)^\dagger Y_\beta(N) Y_\alpha(N) U(\Lambda,0,N)\Psi_{11}(N)|O(N)\rangle}{\langle O(N)|\Psi(N)^\dagger\Psi(N)|O(N)\rangle} \\
=\;& \frac{-2(N-1)\int_\Omega d\Gamma(\boldsymbol{l})\ \int_{\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-\theta_{11}(\boldsymbol{l})+\theta_{11}(\boldsymbol{l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{\Lambda^{-1}l})Z(\boldsymbol{\Lambda^{-1}l}')}{1+(N-1)\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|\psi_{-+}(\boldsymbol{k},\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}')}.
\end{aligned} \tag{578}$$

Comparing this to (550) we find that the vacuum probability density may have an influence on the correlation of the detectors' outcome.

For the Bell state corresponding to the $\Psi_{12}(N)$ field operator (460), using the condition on the field and the polarization angle (459), we get

$$\begin{aligned}
&\langle O(N)|\Psi_{12}(N)^\dagger U(\Lambda,0,N)^\dagger Y_\beta(N) Y_\alpha(N) U(\Lambda,0,N)\Psi_{12}(N)|O(N)\rangle \\
=\;& \frac{8(N-1)}{N}\int\limits_{\Lambda\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})+2\Theta(\Lambda,\boldsymbol{\Lambda l}')) \\
\times\;& \cos 2(\theta_{12}(\boldsymbol{l})-\theta_{12}(\boldsymbol{l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
+\;& \frac{8(N-1)}{N}\int\limits_{\Lambda\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \sin 2(\beta-\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})+2\Theta(\Lambda,\boldsymbol{\Lambda l}')) \\
\times\;& \sin 2(\theta_{12}(\boldsymbol{l})-\theta_{12}(\boldsymbol{l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
=\;& \frac{8(N-1)}{N}\int\limits_{\Lambda\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})+2\Theta(\Lambda,\boldsymbol{\Lambda l}')-\theta_{12}(\boldsymbol{l})+\theta_{12}(\boldsymbol{l}')) \\
\times\;& |\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
=\;& \frac{8(N-1)}{N}\int\limits_{\Lambda\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-\theta_{12}(\boldsymbol{\Lambda l})+\theta_{12}(\boldsymbol{\Lambda l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \qquad (579) \\
=\;& \frac{8(N-1)}{N}\int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-\theta_{12}(\boldsymbol{l})+\theta_{12}(\boldsymbol{l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{\Lambda^{-1}l})Z(\boldsymbol{\Lambda^{-1}l}').
\end{aligned} \tag{580}$$

Then the normalized relativistic EPR average for $\Psi_{12}(N)$ reads

$$\begin{aligned}
&\frac{\langle O(N)|\Psi_{12}(N)^\dagger U(\Lambda,0,N)^\dagger Y_\beta(N) Y_\alpha(N) U(\Lambda,0,N)\Psi_{12}(N)|O(N)\rangle}{\langle O(N)|\Psi_{12}(N)^\dagger\Psi_{12}(N)|O(N)\rangle} \\
=\;& \frac{2(N-1)\int_\Omega d\Gamma(\boldsymbol{l})\ \int_{\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-\theta_{12}(\boldsymbol{l})+\theta_{12}(\boldsymbol{l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{\Lambda^{-1}l})Z(\boldsymbol{\Lambda^{-1}l}')}{1+(N-1)\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|\psi_{-+}(\boldsymbol{k},\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}')}.
\end{aligned} \tag{581}$$

9.3 Relativistic correlation function for maximally correlated in circular polarization basis states - case 1

For maximally correlated in circular polarization basis field operators let us consider the $\Psi_2(N)$ field operator (464). In a realistic case, when the localization of the photon detector leads to a momentum solid angle spread $\boldsymbol{l} \in \Omega,\ \boldsymbol{l}' \in \Omega'$ and for disjoint detectors $\Omega \cap \Omega' = \emptyset$, the correlation function reads:

$$\begin{aligned}
& \langle O(N)|\Psi_2(N)^\dagger U(\Lambda,0,N)^\dagger Y_\beta(N) Y_\alpha(N) U(\Lambda,0,N)\Psi_2(N)|O(N)\rangle \\
= \ & \frac{8(N-1)}{N}\int\limits_{\Lambda\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta+\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})-2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\Re\left(\bar{\psi}_{++}(\boldsymbol{l},\boldsymbol{l}')\psi_{--}(\boldsymbol{l},\boldsymbol{l}')\right) Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
+ \ & \frac{8(N-1)}{N}\int\limits_{\Lambda\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \sin 2(\beta+\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})-2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\Im\left(\bar{\psi}_{++}(\boldsymbol{l},\boldsymbol{l}')\psi_{--}(\boldsymbol{l},\boldsymbol{l}')\right) Z(\boldsymbol{l})Z(\boldsymbol{l}').
\end{aligned} \tag{582}$$

For the Bell state corresponding to the $\Psi_{12}(N)$ field operator (467) using the condition on the field and the polarization angle (466) we get the following unnormalized relativistic EPR average

$$\begin{aligned}
& \langle O(N)|\Psi_{21}(N)^\dagger U(\Lambda,0,N)^\dagger Y_\beta(N) Y_\alpha(N) U(\Lambda,0,N)\Psi_{21}(N)|O(N)\rangle \\
= \ & \frac{-8(N-1)}{N}\int\limits_{\Lambda\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta+\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})-2\Theta(\Lambda,\boldsymbol{\Lambda l}')) \\
\times \ & \cos 2(\theta_{21}(\boldsymbol{l})+\theta_{21}(\boldsymbol{l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
+ \ & \frac{-8(N-1)}{N}\int\limits_{\Lambda\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \sin 2(\beta+\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})-2\Theta(\Lambda,\boldsymbol{\Lambda l}')) \\
\times \ & \sin 2(\theta_{21}(\boldsymbol{l})+\theta_{21}(\boldsymbol{l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
= \ & \frac{-8(N-1)}{N}\int\limits_{\Lambda\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta+\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})-2\Theta(\Lambda,\boldsymbol{\Lambda l}')-\theta_{21}(\boldsymbol{l})-\theta_{21}(\boldsymbol{l}')) \\
\times \ & |\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
= \ & \frac{-8(N-1)}{N}\int\limits_{\Lambda\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta+\alpha-\theta_{21}(\boldsymbol{\Lambda l})-\theta_{21}(\boldsymbol{\Lambda l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \qquad (583) \\
= \ & \frac{-8(N-1)}{N}\int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta+\alpha-\theta_{21}(\boldsymbol{l})-\theta_{21}(\boldsymbol{l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{\Lambda^{-1} l})Z(\boldsymbol{\Lambda^{-1} l}').
\end{aligned} \tag{584}$$

Then the normalized relativistic EPR average reads

$$\begin{aligned}
& \frac{\langle O(N)|\Psi_{21}(N)^\dagger U(\Lambda,0,N)^\dagger Y_\beta(N) Y_\alpha(N) U(\Lambda,0,N)\Psi_{21}(N)|O(N)\rangle}{\langle O(N)|\Psi_{21}(N)^\dagger \Psi_{21}(N)|O(N)\rangle} \\
= \ & \frac{-2\int_\Omega d\Gamma(\boldsymbol{l})\ \int_{\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-\theta_{21}(\boldsymbol{l})+\theta_{21}(\boldsymbol{l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{\Lambda^{-1} l})Z(\boldsymbol{\Lambda^{-1} l}')}{\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|\psi_{--}(\boldsymbol{k},\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}')}.
\end{aligned} \tag{585}$$

Finally for the Bell state corresponding to the $\Psi_{22}(N)$ field operator (471), using the condition on the field and the polarization function (470), we get

$$
\begin{aligned}
& \langle O(N)|\Psi_{22}(N)^\dagger U(\Lambda,0,N)^\dagger Y_\beta(N) Y_\alpha(N) U(\Lambda,0,N)\Psi_{22}(N)|O(N)\rangle \\
= & \frac{8(N-1)}{N}\int\limits_{\Lambda\Omega} d\Gamma(\boldsymbol{l})\int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\;\cos 2(\beta+\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})-2\Theta(\Lambda,\boldsymbol{\Lambda l}')) \\
\times & \cos 2(\theta_{22}(\boldsymbol{l})+\theta_{22}(\boldsymbol{l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
+ & \frac{8(N-1)}{N}\int\limits_{\Lambda\Omega} d\Gamma(\boldsymbol{l})\int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\;\sin 2(\beta+\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})-2\Theta(\Lambda,\boldsymbol{\Lambda l}')) \\
\times & \sin 2(\theta_{22}(\boldsymbol{l})+\theta_{22}(\boldsymbol{l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
= & \frac{8(N-1)}{N}\int\limits_{\Lambda\Omega} d\Gamma(\boldsymbol{l})\int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\;\cos 2(\beta+\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})-2\Theta(\Lambda,\boldsymbol{\Lambda l}')-\theta_{22}(\boldsymbol{l})-\theta_{22}(\boldsymbol{l}')) \\
\times & |\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
= & \frac{8(N-1)}{N}\int\limits_{\Lambda\Omega} d\Gamma(\boldsymbol{l})\int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\;\cos 2(\beta+\alpha-\theta_{22}(\boldsymbol{\Lambda l})-\theta_{22}(\boldsymbol{\Lambda l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
= & \frac{8(N-1)}{N}\int\limits_{\Omega} d\Gamma(\boldsymbol{l})\int\limits_{\Omega'} d\Gamma(\boldsymbol{l}')\;\cos 2(\beta+\alpha-\theta_{22}(\boldsymbol{l})-\theta_{22}(\boldsymbol{l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{\Lambda^{-1} l})Z(\boldsymbol{\Lambda^{-1} l}'),
\end{aligned}
\tag{586}
$$

and for the normalized relativistic EPR average we get

$$
\begin{aligned}
& \frac{\langle O(N)|\Psi_{22}(N)^\dagger U(\Lambda,0,N)^\dagger Y_\beta(N) Y_\alpha(N) U(\Lambda,0,N)\Psi_{22}(N)|O(N)\rangle}{\langle O(N)|\Psi_{22}(N)^\dagger \Psi_{22}(N)|O(N)\rangle} \\
= & \frac{2\int_\Omega d\Gamma(\boldsymbol{l})\;\int_{\Omega'} d\Gamma(\boldsymbol{l}')\;\cos 2(\beta-\alpha-\theta_{22}(\boldsymbol{l})+\theta_{22}(\boldsymbol{l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{\Lambda^{-1} l})Z(\boldsymbol{\Lambda^{-1} l}')}{\int d\Gamma(\boldsymbol{k}) d\Gamma(\boldsymbol{k}')|\psi_{--}(\boldsymbol{k},\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}')}.
\end{aligned}
\tag{587}
$$

9.4 Relativistic correlation of a two-photon state - case 2

Now within the framework of relativistic quantum field theory, let us consider the Einstein-Podolsky-Rosen (EPR) gedankenexperiment in which measurements on detectors are performed by moving observers, only this time we perform a Lorentz transformation only on Alice's detector, so that both detectors are moving with respect to each other.

Let us first consider the field operator corresponding to a two-photon state (444). Under Lorentz transformation on just one detector we have

$$
\begin{aligned}
& \langle O(N)|\Psi(N)^\dagger Y_\beta(N) U(\Lambda,0,N)^\dagger Y_\alpha(N) U(\Lambda,0,N)\Psi(N)|O(N)\rangle \\
= & 4\sum_{ss'=\pm}\int\limits_{\Omega} d\Gamma(\boldsymbol{l})\int\limits_{\Omega'} d\Gamma(\boldsymbol{l}')\; e^{-2i(s\beta+s'\alpha)} e^{4is'\Theta(\Lambda,\boldsymbol{l}')}\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{\Lambda^{-1} l}')\psi_{-s-s'}(\boldsymbol{l},\boldsymbol{\Lambda^{-1} l}') \\
\times & \langle O(N)|I(\boldsymbol{l},N) I(\boldsymbol{\Lambda^{-1} l}',N)|O(N)\rangle \\
+ & 4\sum_{ss'=\pm}\int\limits_{\Omega} d\Gamma(\boldsymbol{l})\int\limits_{\Omega'} d\Gamma(\boldsymbol{l}')\int d\Gamma(\boldsymbol{k})\; e^{-2i(s\beta-s\alpha)} e^{4is\Theta(\Lambda,\boldsymbol{l}')}\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{k})\psi_{ss'}(\boldsymbol{\Lambda^{-1} l}',\boldsymbol{k})\delta_\Gamma(\boldsymbol{l},\boldsymbol{\Lambda^{-1} l}') \\
\times & \langle O(N)|I(\boldsymbol{k},N) I(\boldsymbol{l},N)|O(N)\rangle.
\end{aligned}
\tag{588}
$$

This formula is derived step by step in appendix (J.5), and in the case of disjoint detectors just one part has contribution. Also from the transformation rule on the fields (482) we get:

$$
\begin{aligned}
& \langle O(N)|\Psi(N)^{\dagger}Y_{\beta}(N)U(\Lambda,0,N)^{\dagger}Y_{\alpha}(N)U(\Lambda,0,N)\Psi(N)|O(N)\rangle \\
= & \ 4\sum_{ss'=\pm}\int_{\Omega} d\Gamma(\boldsymbol{l})\int_{\Omega'} d\Gamma(\boldsymbol{l}')\ e^{-2i(s\beta+s'\alpha)}\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{l}')\psi_{-s-s'}(\boldsymbol{l},\boldsymbol{l}')\times\langle O(N)|I(\boldsymbol{l},N)I(\boldsymbol{\Lambda^{-1}l}',N)|O(N)\rangle \\
= & \ 4\sum_{ss'=\pm}\int_{\Omega} d\Gamma(\boldsymbol{l})\int_{\Omega'} d\Gamma(\boldsymbol{l}')\ e^{-2i(s\beta+s'\alpha)}\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{l}')\psi_{-s-s}(\boldsymbol{l},\boldsymbol{l}') \\
\times & \left(\frac{1}{N}\left(\bar{O}(\boldsymbol{l})O(\boldsymbol{\Lambda^{-1}l}')\delta_{\Gamma}(\boldsymbol{l},\boldsymbol{\Lambda^{-1}l}')+(N-1)Z(\boldsymbol{l})Z(\boldsymbol{\Lambda^{-1}l}')\right)\right) \\
= & \ \frac{4(N-1)}{N}\sum_{ss'=\pm}\int_{\Omega} d\Gamma(\boldsymbol{l})\int_{\Omega'} d\Gamma(\boldsymbol{l}')\ e^{-2i(s\beta+s'\alpha)}\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{l}')\psi_{-s-s'}(\boldsymbol{l},\boldsymbol{l}')Z(\boldsymbol{l})Z(\boldsymbol{\Lambda^{-1}l}') \\
= & \ \frac{4(N-1)}{N}\sum_{ss'=\pm}\int_{\Omega} d\Gamma(\boldsymbol{l})\int_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ e^{-2i(s\beta+s'\alpha)}\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{\Lambda l}')\psi_{-s-s'}(\boldsymbol{l},\boldsymbol{\Lambda l}')Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
= & \ \frac{4(N-1)}{N}\sum_{ss'=\pm}\int_{\Omega} d\Gamma(\boldsymbol{l})\int_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ e^{-2i(s\beta+s'\alpha)}e^{4i(s'\Theta(\Lambda,\boldsymbol{\Lambda l}'))}\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{l}')\psi_{-s-s'}(\boldsymbol{l},\boldsymbol{l}')Z(\boldsymbol{l})Z(\boldsymbol{l}').
\end{aligned}
\tag{589}
$$

Now let us take a closer look at part:

$$
\begin{aligned}
& \sum_{ss'=\pm} e^{-2i(s\beta+s'\alpha)}e^{4i(s'\Theta(\Lambda,\boldsymbol{\Lambda l}'))}\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{l}')\psi_{-s-s'}(\boldsymbol{l},\boldsymbol{l}') \\
= & \ 2\cos 2(\beta+\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\Re\left(\bar{\psi}_{++}(\boldsymbol{l},\boldsymbol{l}')\psi_{--}(\boldsymbol{l},\boldsymbol{l}')\right) \\
+ & \ 2\sin 2(\beta+\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\Im\left(\bar{\psi}_{++}(\boldsymbol{l},\boldsymbol{l}')\psi_{--}(\boldsymbol{l},\boldsymbol{l}')\right) \\
+ & \ 2\cos 2(\beta-\alpha+2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\Re\left(\bar{\psi}_{+-}(\boldsymbol{l},\boldsymbol{l}')\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')\right) \\
+ & \ 2\sin 2(\beta-\alpha+2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\Im\left(\bar{\psi}_{+-}(\boldsymbol{l},\boldsymbol{l}')\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')\right).
\end{aligned}
\tag{590}
$$

Then the unnormalized relativistic EPR average for a two-photon state, in a situation when a Lorentz transformation is performed only on Alice's detector, can be written in the form

$$
\begin{aligned}
& \langle O(N)|\Psi(N)^{\dagger}Y_{\beta}(N)U(\Lambda,0,N)^{\dagger}Y_{\alpha}(N)U(\Lambda,0,N)\Psi(N)|O(N)\rangle \\
= & \ \frac{8(N-1)}{N}\int_{\Omega} d\Gamma(\boldsymbol{l})\int_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta+\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\Re\left(\bar{\psi}_{++}(\boldsymbol{l},\boldsymbol{l}')\psi_{--}(\boldsymbol{l},\boldsymbol{l}')\right)Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
+ & \ \frac{8(N-1)}{N}\int_{\Omega} d\Gamma(\boldsymbol{l})\int_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \sin 2(\beta+\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\Im\left(\bar{\psi}_{++}(\boldsymbol{l},\boldsymbol{l}')\psi_{--}(\boldsymbol{l},\boldsymbol{l}')\right)Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
+ & \ \frac{8(N-1)}{N}\int_{\Omega} d\Gamma(\boldsymbol{l})\int_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha+2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\Re\left(\bar{\psi}_{+-}(\boldsymbol{l},\boldsymbol{l}')\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')\right)Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
+ & \ \frac{8(N-1)}{N}\int_{\Omega} d\Gamma(\boldsymbol{l})\int_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \sin 2(\beta-\alpha+2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\Im\left(\bar{\psi}_{+-}(\boldsymbol{l},\boldsymbol{l}')\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')\right)Z(\boldsymbol{l})Z(\boldsymbol{l}').
\end{aligned}
\tag{591}
$$

9.5 Relativistic correlation function for maximally anti-correlated in circular polarization basis states - case 2

Now, taking results form the previous section, we will consider relativistic correlation functions for the four Bell states. Let us start from maximally anti-correlated states in circular polarization basis corresponding to the field operator $\Psi_1(N)$. In a realistic case when the localization of the photon detector leads to a momentum solid angle spread $\boldsymbol{l} \in \Omega,\ \boldsymbol{l}' \in \Omega'$ and disjoint detectors $\Omega \cap \Omega' = \emptyset$, the correlation function reads:

$$\begin{aligned}
&\langle O(N)|\Psi_1(N)^\dagger Y_\beta(N)U(\Lambda,0,N)^\dagger Y_\alpha(N)U(\Lambda,0,N)\Psi_1(N)|O(N)\rangle \\
= \ & \frac{8(N-1)}{N}\int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha+2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\Re\left(\bar{\psi}_{+-}(\boldsymbol{l},\boldsymbol{l}')\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')\right) Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
+ \ & \frac{8(N-1)}{N}\int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \sin 2(\beta-\alpha+2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\Im\left(\bar{\psi}_{+-}(\boldsymbol{l},\boldsymbol{l}')\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')\right) Z(\boldsymbol{l})Z(\boldsymbol{l}').
\end{aligned} \tag{592}$$

For the Bell state corresponding to the $\Psi_{11}(N)$ field operator (456), using condition (455), we get an unnormalized relativistic EPR average of the form

$$\begin{aligned}
&\langle O(N)|\Psi_{11}(N)^\dagger Y_\beta(N)U(\Lambda,0,N)^\dagger Y_\alpha(N)U(\Lambda,0,N)\Psi_{11}(N)|O(N)\rangle \\
= \ & \frac{-8(N-1)}{N}\int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha+2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\cos 2(\theta_{11}(\boldsymbol{l})-\theta_{11}(\boldsymbol{l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
+ \ & \frac{-8(N-1)}{N}\int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \sin 2(\beta-\alpha+2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\sin 2(\theta_{11}(\boldsymbol{l})-\theta_{11}(\boldsymbol{l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
= \ & \frac{-8(N-1)}{N}\int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha+2\Theta(\Lambda,\boldsymbol{\Lambda l}')-\theta_{11}(\boldsymbol{l})+\theta_{11}(\boldsymbol{l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
= \ & \frac{-8(N-1)}{N}\int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-\theta_{11}(\boldsymbol{l})+\theta_{11}(\boldsymbol{\Lambda l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \qquad (593) \\
= \ & \frac{-8(N-1)}{N}\int\limits_{\Lambda^{-1}\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-\theta_{11}(\boldsymbol{\Lambda^{-1} l})+\theta_{11}(\boldsymbol{l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{\Lambda^{-1} l})Z(\boldsymbol{\Lambda^{-1} l}').
\end{aligned} \tag{594}$$

Then the normalized relativistic EPR average for $\Psi_{11}(N)$ reads

$$\begin{aligned}
&\frac{\langle O(N)|\Psi_{11}(N)^\dagger Y_\beta(N)U(\Lambda,0,N)^\dagger Y_\alpha(N)U(\Lambda,0,N)\Psi_{11}(N)|O(N)\rangle}{\langle O(N)|\Psi(N)^\dagger\Psi(N)|O(N)\rangle} \\
= \ & \frac{-2(N-1)\int_\Omega d\Gamma(\boldsymbol{l})\ \int_{\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-\theta_{11}(\boldsymbol{l})+\theta_{11}(\boldsymbol{l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{\Lambda^{-1} l}')}{1+(N-1)\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|\psi_{-+}(\boldsymbol{k},\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}')}.
\end{aligned} \tag{595}$$

Next for the Bell state corresponding to the $\Psi_{12}(N)$ field operator (460), and using condition (459), we get

$$
\begin{aligned}
&\langle O(N)|\Psi_{12}(N)^\dagger Y_\beta(N)U(\Lambda,0,N)^\dagger Y_\alpha(N)U(\Lambda,0,N)\Psi_{12}(N)|O(N)\rangle \\
= &\ \frac{8(N-1)}{N}\int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha+2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\cos 2(\theta_{12}(\boldsymbol{l})-\theta_{12}(\boldsymbol{l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
+ &\ \frac{8(N-1)}{N}\int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \sin 2(\beta-\alpha+2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\sin 2(\theta_{12}(\boldsymbol{l})-\theta_{12}(\boldsymbol{l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
= &\ \frac{8(N-1)}{N}\int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha+2\Theta(\Lambda,\boldsymbol{\Lambda l}')-\theta_{12}(\boldsymbol{l})+\theta_{12}(\boldsymbol{l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
= &\ \frac{8(N-1)}{N}\int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-\theta_{12}(\boldsymbol{l})+\theta_{12}(\boldsymbol{\Lambda l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \qquad (596) \\
= &\ \frac{8(N-1)}{N}\int\limits_{\Lambda^{-1}\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-\theta_{12}(\boldsymbol{\Lambda^{-1}l})+\theta_{12}(\boldsymbol{l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{\Lambda^{-1}l})Z(\boldsymbol{\Lambda^{-1}l}'),
\end{aligned}
$$

(597)

and the normalized relativistic EPR average for $\Psi_{12}(N)$ reads

$$
\begin{aligned}
&\frac{\langle O(N)|\Psi_{12}(N)^\dagger Y_\beta(N)U(\Lambda,0,N)^\dagger Y_\alpha(N)U(\Lambda,0,N)\Psi_{12}(N)|O(N)\rangle}{\langle O(N)|\Psi_{12}(N)^\dagger\Psi_{12}(N)|O(N)\rangle} \\
= &\ \frac{2(N-1)\int_\Omega d\Gamma(\boldsymbol{l})\ \int_{\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-\theta_{12}(\boldsymbol{l})+\theta_{12}(\boldsymbol{l}'))|\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{\Lambda^{-1}l}')}{1+(N-1)\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|\psi_{-+}(\boldsymbol{k},\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}')}.
\end{aligned}
$$

(598)

9.6 Relativistic correlation function for maximally correlated in circular polarization basis states - case 2

For maximally correlated in circular polarization basis field operators let us consider $\Psi_2(N)$. In a realistic case when the localization of the photon detector leads to a momentum solid angle spread $\boldsymbol{l}\in\Omega,\ \boldsymbol{l}'\in\Omega'$ and disjoint detectors $\Omega\cap\Omega'=\emptyset$: the correlation function reads:

$$
\begin{aligned}
&\langle O(N)|\Psi_{2}(N)^\dagger Y_\beta(N)U(\Lambda,0,N)^\dagger Y_\alpha(N)U(\Lambda,0,N)\Psi_{2}(N)|O(N)\rangle \\
= &\ \frac{8(N-1)}{N}\int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta+\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\Re\left(\bar\psi_{++}(\boldsymbol{l},\boldsymbol{l}')\psi_{--}(\boldsymbol{l},\boldsymbol{l}')\right) Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
+ &\ \frac{8(N-1)}{N}\int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \sin 2(\beta+\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\Im\left(\bar\psi_{++}(\boldsymbol{l},\boldsymbol{l}')\psi_{--}(\boldsymbol{l},\boldsymbol{l}')\right) Z(\boldsymbol{l})Z(\boldsymbol{l}'). \qquad (599)
\end{aligned}
$$

Now for the Bell state corresponding to the $\Psi_{21}(N)$ field operator (467), using the condition (466), we get the following unnormalized EPR average

$$\begin{aligned}
&\langle O(N)|\Psi_{21}(N)^\dagger Y_\beta(N)U(\Lambda,0,N)^\dagger Y_\alpha(N)U(\Lambda,0,N)\Psi_{21}(N)|O(N)\rangle \\
=\ & \frac{-8(N-1)}{N}\int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta+\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\cos 2(\theta_{21}(\boldsymbol{l})+\theta_{21}(\boldsymbol{l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
+\ & \frac{-8(N-1)}{N}\int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \sin 2(\beta+\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\sin 2(\theta_{21}(\boldsymbol{l})+\theta_{21}(\boldsymbol{l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
=\ & \frac{-8(N-1)}{N}\int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta+\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l}')-\theta_{21}(\boldsymbol{l})-\theta_{21}(\boldsymbol{l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
=\ & \frac{-8(N-1)}{N}\int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta+\alpha-\theta_{21}(\boldsymbol{l})-\theta_{21}(\boldsymbol{\Lambda l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \qquad (600) \\
=\ & \frac{-8(N-1)}{N}\int\limits_{\Lambda^{-1}\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta+\alpha-\theta_{21}(\boldsymbol{\Lambda^{-1}l})-\theta_{21}(\boldsymbol{l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{\Lambda^{-1}l})Z(\boldsymbol{\Lambda^{-1}l}').
\end{aligned} \tag{601}$$

Then the relativistic normalized EPR average reads

$$\begin{aligned}
&\frac{\langle O(N)|\Psi_{21}(N)^\dagger Y_\beta(N)U(\Lambda,0,N)^\dagger Y_\alpha(N)U(\Lambda,0,N)\Psi_{21}(N)|O(N)\rangle}{\langle O(N)|\Psi_{21}(N)^\dagger\Psi_{21}(N)|O(N)\rangle} \\
=\ & \frac{-2\int_\Omega d\Gamma(\boldsymbol{l})\ \int_{\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-\theta_{21}(\boldsymbol{l})+\theta_{21}(\boldsymbol{l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{\Lambda^{-1}l}')}{\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|\psi_{--}(\boldsymbol{k},\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}')}.
\end{aligned} \tag{602}$$

Finally for the Bell state corresponding to the $\Psi_{22}(N)$ field operator (471), using the condition (470), we get

$$\begin{aligned}
&\langle O(N)|\Psi_{22}(N)^\dagger Y_\beta(N)U(\Lambda,0,N)^\dagger Y_\alpha(N)U(\Lambda,0,N)\Psi_{22}(N)|O(N)\rangle \\
=\ & \frac{8(N-1)}{N}\int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta+\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\cos 2(\theta_{22}(\boldsymbol{l})+\theta_{22}(\boldsymbol{l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
+\ & \frac{8(N-1)}{N}\int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \sin 2(\beta+\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l}'))\sin 2(\theta_{22}(\boldsymbol{l})+\theta_{22}(\boldsymbol{l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
=\ & \frac{8(N-1)}{N}\int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta+\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l}')-\theta_{22}(\boldsymbol{l})-\theta_{22}(\boldsymbol{l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \\
=\ & \frac{8(N-1)}{N}\int\limits_{\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Lambda\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta+\alpha-\theta_{22}(\boldsymbol{l})-\theta_{22}(\boldsymbol{\Lambda l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{l}') \qquad (603) \\
=\ & \frac{8(N-1)}{N}\int\limits_{\Lambda^{-1}\Omega} d\Gamma(\boldsymbol{l}) \int\limits_{\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta+\alpha-\theta_{22}(\boldsymbol{\Lambda^{-1}l})-\theta_{22}(\boldsymbol{l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{\Lambda^{-1}l})Z(\boldsymbol{\Lambda^{-1}l}'),
\end{aligned} \tag{604}$$

and for the normalized relativistic EPR average we get

$$\begin{aligned}
&\frac{\langle O(N)|\Psi_{22}(N)^\dagger Y_\beta(N)U(\Lambda,0,N)^\dagger Y_\alpha(N)U(\Lambda,0,N)\Psi_{22}(N)|O(N)\rangle}{\langle O(N)|\Psi_{22}(N)^\dagger\Psi_{22}(N)|O(N)\rangle} \\
=\ & \frac{2\int_\Omega d\Gamma(\boldsymbol{l})\ \int_{\Omega'} d\Gamma(\boldsymbol{l}')\ \cos 2(\beta-\alpha-\theta_{22}(\boldsymbol{l})+\theta_{22}(\boldsymbol{l}'))|\psi_{--}(\boldsymbol{l},\boldsymbol{l}')|^2 Z(\boldsymbol{l})Z(\boldsymbol{\Lambda^{-1}l}')}{\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|\psi_{--}(\boldsymbol{k},\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}')}.
\end{aligned} \tag{605}$$

9.7 Results and conclusions

This chapter contains new results. Relativistic EPR averages where calculated for all four Bell states. Two cases where considered here: where two detectors are transformed under Lorentz transformation in such a way that the still maintain in the same reference frame and where just one of the detectors is transformed under Lorentz transformation. The main conclusion from this chapter is that there may be a relativistic effect on the degree of violation in EPR-type experiment. One may look at the formulas derived for the relativistic EPR-type correlations as the transformation on the polarization angle and detectors angle spread or a transformation on the vacuum probability function. Assuming a non-unique vacuum, the vacuum probability function $Z(\boldsymbol{k})$ may have an impact on the detectors outcome.

10 Final results and conclusions

The main motivation for this work was to take a closer look at a relativistic model for boson fields in reducible representations of harmonic oscillator Lie algebras (HOLA) proposed by Czachor [6]-[19] with an application to relativistic EPR correlations. This work resulted in conclusions relating to the model of a four-dimensional polarization space, covariance of the potential operator or a model of a covariant two-photon field. On the other hand, employing reducible representations for relativistic EPR-type experiments may show the role that the oscillator number N and vacuum probability density $Z(\boldsymbol{k})$, known from such representations, play in this model.

Most of the results from chapter 2 were presented in [6] - [13] and lecture notes [19]. One new remark has been made here, i.e. it has been shown that reducible representations taken within the whole frequency spectrum hold the "standard theory" harmonic oscillator Lie algebras.

From chapter 3 come new results. Here a construction for the four-dimensional polarization space coming from a definition of the covariant Hamiltonian (108) is presented. Further analysis is done for such formalism, especially regarding the interpretation of the ladder operators for the time-like degree of freedom a_0. Strong arguments are given in favor of an interpretation in which the operator annihilating vacuum is a raising energy operator. Such an interpretation gives a non divergent vacuum representation and positive scalar products. These results are in agreement the the four-dimensional quantization of the potential operator in Czachor and Naudts [12], and Czachor and Wrzask [13] papers.

Further in chapter 4 N-oscillator reducible representations for the four-dimensional polarization are presented. Using the covariant Hamiltonian (194) for N-oscillator representation, we find out that such a formalism is free from vacuum energy divergences. The convergence of vacuum energy is guaranteed by a proper choice of the vacuum probability density function $Z(\boldsymbol{k})$ and the N parameter may even be a finite number. Further sections 4.4 and 4.8 of this chapter contain new results, showing the existence of Ψ_{EM} vectors, which reproduce standard electromagnetic fields (i.e. photons with two polarization degrees of freedom) from the four-dimensional covariant formalism (i.e. with two additional longitudinal and time-like). It is interesting that such vectors have a coherent-like structure.

Most of the results form chapter 5, where already presented in Czachor and Naudts [12], and Czachor and Wrzask [13]. The notation, is set differently here, in a way that the homogeneous Lorentz and gauge transformations are treated as separate non-commuting transformations. Further generators of these transformations coming from the canonical variables are shown. In next sections the composition law for homogeneous Lorentz transformation and the additivity of Lorentz transformation on the gauge parameter are proved. These are new results. On next sections of this chapter the homogeneous Lorentz transformation acting on: the four-vector potential, the electromagnetic field operator and vacuum are shown. As a by product of the Lorentz transformation on a non-unique vacuum we observe that the vacuum field transforms as a scalar field and this may have its consequence in relativistic EPR-type experiments. In 5.11 it has been pointed out that for the reducible covariant representation there exist a transformation on the spin-frame level that corresponds to a gauge transformation on the potential level for Ψ_{EM} vectors introduced earlier. In section 5.12 invariants of the Lorentz transformation are shown. Let us stress that the "ghost operator" coming from the two extra degrees of photon polarization is an invariant.

All further chapters contain new results. In chapter 6 it has been shown that it is possible to model Bell states in quantum field theory background of N-oscillator reducible representations. The main assumption is that Bell states are maximally correlated or maximally anti-correlated in two polarization bases: circular and linear. However it should be stressed here that in this model the linear polarization angles are dependent on momentum, and from the condition for maximal correlation in both bases we get conditions on the fields and polarization angle functions (455), (459), (466) and (470). Employing such momentum dependent polarization angles is important for maintaining Lorentz covariance in both bases.

In chapter 7 it has been shown that theoretically it is possible to maintain Lorentz invariance of the field operators corresponding to the four Bell states introduced earlier in both polarization bases. The conclusion is: to obtain maximal correlation for EPR-type experiments in both bases one has to employ momentum dependent polarization functions that transform under Lorentz transformation in such a way that they compensate the Wigner phase $2\Theta(\Lambda, \boldsymbol{k})$.

In chapter 8 the EPR correlation functions, for all four Bell states where calculated in reducible representations. First conclusion involves the N parameter. In reducible representations the N parameter does not necessary have to go to infinity, since each oscillator is a superposition of already infinitely many different momentum states. If we made an assumption that the N parameter is a finite large number, it would have had an influence on the outcome of the EPR average for states maximally anti-correlated in circular basis. Like shown in section 8.3 the EPR average for maximally anti-correlated in circular polarization basis states depends on the N parameter. The extra term in the denominator of the EPR averages for Bell states corresponding to the $\Psi_1(N)$ field operator may have influence on the outcome compared with the Bell states corresponding to the field operator $\Psi_2(N)$. If any experiments confirmed a smaller outcome of the EPR average for maximally anti-correlated in circular basis states comparing with maximally correlated in circular basis states, it could have spoken in favor for the N parameter being a finite number. On the other hand for the limit $N \to \infty$ the correlation function in reducible representation does not show difference from the irreducible representation except from the momentum dependent polarization angle shift phase. Other than that, for $Z(\boldsymbol{k})$ being flat in the detectors momentum solid angle spread, it is hard to distinguish such representation from "standard models".

Finally from chapter 9 the main conclusion is that there may be a relativistic effect on the degree of violation in EPR-type experiments for photon fields. Two cases where considered here: where the two detectors are transformed under Lorentz transformation in such a way that they still maintain in the same reference frame and where just one of the detectors is transformed under Lorentz transformation. Employing a model for relativistic EPR-type experiments in reducible representation may show the role that the vacuum probability density function $Z(\boldsymbol{k})$ plays in such relativistic experiments. It turns out that assuming such non-unique vacuum, the vacuum probability density function $Z(\boldsymbol{k})$ may have an impact on the detectors outcome for such relativistic model.

To see this let us schematically rewrite the results presented in previous sections. For two detectors Y_β, Y_α and for some two-photon field Ψ operator we will write the EPR average

$$\langle O|\Psi^\dagger Y_\beta Y_\alpha \Psi|O\rangle. \tag{606}$$

We may use the unitarity of the Lorentz transformation and assuming an invariant two-photon field operator, i.e. $U_\Lambda^\dagger \Psi U_\Lambda = \Psi$, we have

$$\langle O|\Psi^\dagger Y_\beta Y_\alpha \Psi|O\rangle = \langle O|U_\Lambda U_\Lambda^\dagger \Psi^\dagger U_\Lambda U_\Lambda^\dagger Y_\beta U_\Lambda U_\Lambda^\dagger Y_\alpha U_\Lambda U_\Lambda^\dagger \Psi U_\Lambda U_\Lambda^\dagger|O\rangle = \langle O_{\Lambda^{-1}}|\Psi^\dagger Y_{\Lambda\beta} Y_{\Lambda\alpha}\Psi|O_{\Lambda^{-1}}\rangle. \tag{607}$$

This means that performing a transformation on the detectors $Y_{\Lambda\beta}Y_{\Lambda\alpha}$ and a compensating transformation on the vacuum field, denoted here by $O_{\Lambda^{-1}}$, would be equivalent to not performing any transformation at all. But when we perform a Lorentz transformation like in the first case, i.e. on both detectors in such a way that they remain in the same reference frame, the EPR average takes the form

$$\langle O|\Psi^\dagger U_\Lambda^\dagger Y_\beta Y_\alpha U_\Lambda \Psi|O\rangle = \langle O|\Psi^\dagger U_\Lambda^\dagger Y_\beta U_\Lambda U_\Lambda^\dagger Y_\alpha U_\Lambda \Psi|O\rangle = \langle O|\Psi^\dagger Y_{\Lambda\beta} Y_{\Lambda\alpha}\Psi|O\rangle. \tag{608}$$

On the other hand we may perform an unitary transformation to on states, only we have to remember that in this model we assume a non-unique vacuum and invariant field operators, i.e.

$$\langle O|\Psi^\dagger U_\Lambda^\dagger Y_\beta Y_\alpha U_\Lambda \Psi|O\rangle = \langle O|U_\Lambda^\dagger U_\Lambda \Psi^\dagger U_\Lambda^\dagger Y_\beta Y_\alpha U_\Lambda \Psi U_\Lambda^\dagger U_\Lambda|O\rangle = \langle O_\Lambda|\Psi^\dagger Y_\beta Y_\alpha \Psi|O_\Lambda\rangle. \tag{609}$$

This means that performing a transformation on both detectors is equivalent to performing a transformation on the vacuum which is denoted by O_Λ. Now for the second case experiment we perform a Lorentz transformation only on Alice's detector, i.e.

$$\begin{aligned}\langle O|\Psi^\dagger Y_\beta U_\Lambda^\dagger Y_\alpha U_\Lambda \Psi|O\rangle &= \langle O|\Psi^\dagger Y_\beta Y_{\Lambda\alpha}\Psi|O\rangle = \langle O|U_\Lambda^\dagger U_\Lambda \Psi^\dagger U_\Lambda^\dagger U_\Lambda Y_\beta U_\Lambda^\dagger Y_\alpha U_\Lambda \Psi U_\Lambda^\dagger U_\Lambda|O\rangle \\ &= \langle O_\Lambda|\Psi^\dagger Y_{\Lambda^{-1}\beta} Y_\alpha \Psi|O_\Lambda\rangle.\end{aligned} \tag{610}$$

As we can see, performing a Lorentz transformation on Alice's detector does not have to result in the same EPR average as performing an inverse Lorentz transformation on Bob's detector. If any experiments confirmed such results, this could have spoken in favor for a non-unique vacuum representation and its impact on such relativistic experiments.

Appendices

A Spinors and tetrads

Following the notation convention used by Penrose and Rindler [1], we will skip the bars in complex conjugation:

$$\omega^{A'} := \bar{\omega}^{A'}, \qquad \pi^{A'} := \bar{\pi}^{A'}, \tag{A.1}$$

$$\overline{\omega^{A}} = \omega^{A'} = {\varepsilon_{0'}}^{A'}, \qquad \overline{\pi^{A}} = \pi^{A'} = {\varepsilon_{1'}}^{A'}, \tag{A.2}$$

$$\omega_{A'}\pi^{A'} = 1, \qquad \pi_{A'}\omega^{A'} = -1. \tag{A.3}$$

Then the null tetrad with respect to the spin-frame can be written in the form

$$k_a(\boldsymbol{k}) = \pi_A(\boldsymbol{k})\pi_{A'}(\boldsymbol{k}), \tag{A.4}$$

$$\omega_a(\boldsymbol{k}) = \omega_A(\boldsymbol{k})\omega_{A'}(\boldsymbol{k}), \tag{A.5}$$

$$m_a(\boldsymbol{k}) = \omega_A(\boldsymbol{k})\pi_{A'}(\boldsymbol{k}), \tag{A.6}$$

$$\bar{m}_a(\boldsymbol{k}) = \pi_A(\boldsymbol{k})\omega_{A'}(\boldsymbol{k}). \tag{A.7}$$

Then the Minkowski tetrad may be written with respect to the null tetrad

$$x_a(\boldsymbol{k}) = \frac{1}{\sqrt{2}}\left(m_a(\boldsymbol{k}) + \bar{m}_a(\boldsymbol{k})\right), \tag{A.8}$$

$$y_a(\boldsymbol{k}) = \frac{i}{\sqrt{2}}\left(m_a(\boldsymbol{k}) - \bar{m}_a(\boldsymbol{k})\right), \tag{A.9}$$

$$z_a(\boldsymbol{k}) = \frac{1}{\sqrt{2}}\left(\omega_a(\boldsymbol{k}) + k_a(\boldsymbol{k})\right), \tag{A.10}$$

$$t_a(\boldsymbol{k}) = \frac{1}{\sqrt{2}}\left(\omega_a(\boldsymbol{k}) - k_a(\boldsymbol{k})\right), \tag{A.11}$$

These formulas can be easily inverted to give

$$m_a(\boldsymbol{k}) = \frac{1}{\sqrt{2}}\left(x_a(\boldsymbol{k}) - iy_a(\boldsymbol{k})\right), \tag{A.12}$$

$$\bar{m}_a(\boldsymbol{k}) = \frac{1}{\sqrt{2}}\left(x_a(\boldsymbol{k}) + iy_a(\boldsymbol{k})\right), \tag{A.13}$$

$$\omega_a(\boldsymbol{k}) = \frac{1}{\sqrt{2}}\left(t_a(\boldsymbol{k}) + z_a(\boldsymbol{k})\right), \tag{A.14}$$

$$k_a(\boldsymbol{k}) = \frac{1}{\sqrt{2}}\left(t_a(\boldsymbol{k}) - z_a(\boldsymbol{k})\right). \tag{A.15}$$

Furthermore,

$$t_a(\boldsymbol{k})t^a(\boldsymbol{k}) = 1, \qquad x_a(\boldsymbol{k})x^a(\boldsymbol{k}) = -1, \qquad y_a(\boldsymbol{k})y^a(\boldsymbol{k}) = -1, \qquad z_a(\boldsymbol{k})z^a(\boldsymbol{k}) = -1. \tag{A.16}$$

$$k_a(\boldsymbol{k})\omega^a(\boldsymbol{k}) = 1, \qquad m_a(\boldsymbol{k})\bar{m}^a(\boldsymbol{k}) = -1, \tag{A.17}$$

$$k_a(\boldsymbol{k})x^a(\boldsymbol{k}) = k_a(\boldsymbol{k})y^a(\boldsymbol{k}) = 0, \qquad k_a(\boldsymbol{k})z^a(\boldsymbol{k}) = k_a(\boldsymbol{k})t^a(\boldsymbol{k}) = \frac{1}{\sqrt{2}}. \tag{A.18}$$

The following antisymmetric structures of tetrads are derived here for the electromagnetic field operator (252):

$$x_a k_b - k_a x_b = \frac{1}{\sqrt{2}}\left((\omega_A\pi_{A'} + \pi_A\omega_{A'})\pi_B\bar{\pi}_{B'} - \pi_A\bar{\pi}_{A'}(\omega_B\pi_{B'} + \pi_B\omega_{B'})\right) = \frac{1}{\sqrt{2}}(\varepsilon_{AB}\pi_{A'}\pi_{B'} + \varepsilon_{A'B'}\pi_A\pi_B), \tag{A.19}$$

$$y_a k_b - k_a y_b = \frac{i}{\sqrt{2}}\left((\omega_A\pi_{A'} - \pi_A\omega_{A'})\pi_B\pi_{B'} - \pi_A\pi_{A'}(\omega_B\pi_{B'} - \pi_B\omega_{B'})\right) = \frac{i}{\sqrt{2}}(\varepsilon_{AB}\pi_{A'}\pi_{B'} - \bar{\varepsilon}_{A'B'}\pi_A\pi_B), \tag{A.20}$$

$$z_a k_b - k_a z_b = \frac{1}{\sqrt{2}}\left((\omega_A\omega_{A'} - \pi_A\pi_{A'})\pi_B\pi_{B'} - \pi_A\pi_{A'}(\omega_B\omega_{B'} - \pi_B\pi_{B'})\right) = \frac{1}{\sqrt{2}}(\omega_A\omega_{A'}\pi_B\pi_{B'} - \pi_A\pi_{A'}\omega_B\omega_{B'}), \tag{A.21}$$

$$t_a k_b - k_a t_b = \frac{1}{\sqrt{2}}\left((\omega_A\omega_{A'} + \pi_A\pi_{A'})\pi_B\pi_{B'} - \pi_A\pi_{A'}(\omega_B\omega_{B'} + \pi_B\pi_{B'})\right) = \frac{1}{\sqrt{2}}(\omega_A\omega_{A'}\pi_B\pi_{B'} - \pi_A\pi_{A'}\omega_B\omega_{B'}). \tag{A.22}$$

B Spin-frame Lorentz transformation

The Lorentz transformation rule for any covariant and contravariant spinor holds

$$\begin{aligned}\psi_A(\boldsymbol{k}) &\mapsto \Lambda\psi_A(\boldsymbol{k}) = {\Lambda_A}^B\psi_B(\boldsymbol{\Lambda^{-1}k}), && \text{(B.1)}\\ \psi^A(\boldsymbol{k}) &\mapsto \Lambda\psi^A(\boldsymbol{k}) = \psi^B(\boldsymbol{\Lambda^{-1}k}){\Lambda^{-1}}_B{}^A = -{\Lambda^A}_B\psi^B(\boldsymbol{\Lambda^{-1}k}). && \text{(B.2)}\end{aligned}$$

Here $\boldsymbol{\Lambda^{-1}k}$ is a space-like component of a four-vector ${\Lambda^{-1}}_a{}^b k_b(\boldsymbol{k})$.

Some transformation rules of spinor fields

$$e^{-i\Theta(\Lambda,k)} = \omega_A(\boldsymbol{k})\Lambda\pi^A(\boldsymbol{k}), \tag{B.3}$$

$$e^{i\Theta(\Lambda,k)} = \omega_{A'}(\boldsymbol{k})\Lambda\pi^{A'}(\boldsymbol{k}), \tag{B.4}$$

$$\pi_A(\boldsymbol{k}) = e^{i\Theta(\Lambda,\boldsymbol{k})}{\Lambda_A}^B\pi_B(\boldsymbol{\Lambda^{-1}k}), \qquad \omega_A(\boldsymbol{k}) = e^{-i\Theta(\Lambda,\boldsymbol{k})}{\Lambda_A}^B\omega_B(\boldsymbol{\Lambda^{-1}k}), \tag{B.5}$$

$$\pi_A(\boldsymbol{\Lambda k}) = e^{i\Theta(\Lambda,\boldsymbol{\Lambda k})}{\Lambda_A}^B\pi_B(\boldsymbol{k}), \qquad \omega_A(\boldsymbol{\Lambda k}) = e^{-i\Theta(\Lambda,\boldsymbol{\Lambda k})}{\Lambda_A}^B\omega_B(\boldsymbol{k}), \tag{B.6}$$

$$m_a(\boldsymbol{\Lambda k})e^{2i\Theta(\Lambda,\boldsymbol{\Lambda k})} = {\Lambda_a}^b m_b(\boldsymbol{k}), \qquad \bar{m}_a(\boldsymbol{\Lambda k})e^{-2i\Theta(\Lambda,\boldsymbol{\Lambda k})} = {\Lambda_a}^b\bar{m}_b(\boldsymbol{k}), \tag{B.7}$$

$$z_a(\boldsymbol{\Lambda k}) = {\Lambda_a}^b z_b(\boldsymbol{k}), \qquad t_a(\boldsymbol{\Lambda k}) = {\Lambda_a}^b t_b(\boldsymbol{k}), \tag{B.8}$$

$$k_a(\boldsymbol{\Lambda k}) = {\Lambda_a}^b k_b(\boldsymbol{k}). \tag{B.9}$$

Here $\boldsymbol{\Lambda k}$ is a space like part of a four-vector ${\Lambda_a}^b k_b(\boldsymbol{k})$ and ${\Lambda_A}^B$ is denoted as an unprimed SL(2,C) matrix corresponding to the ${\Lambda_a}^b \in SO(1,3)$. Also may be useful

$$\begin{aligned} e^{-i\Theta(\Lambda,k)}\pi_A(\boldsymbol{k}) &= {\Lambda_A}^B\pi_B(\boldsymbol{\Lambda^{-1}k}) = \Lambda\pi_A(\boldsymbol{k}), && \text{(B.10)}\\ e^{-i\Theta(\Lambda,k)}\omega_{A'}(\boldsymbol{k}) &= {\Lambda_{A'}}^{B'}\omega_{B'}(\boldsymbol{\Lambda^{-1}k}) = \Lambda\omega_{A'}(\boldsymbol{k}), && \text{(B.11)}\\ e^{i\Theta(\Lambda,k)}\pi_{A'}(\boldsymbol{k}) &= {\Lambda_{A'}}^{B'}\pi_{B'}(\boldsymbol{\Lambda^{-1}k}) = \Lambda\pi_{A'}(\boldsymbol{k}), && \text{(B.12)}\\ e^{i\Theta(\Lambda,k)}\omega_A(\boldsymbol{k}) &= {\Lambda_A}^B\omega_B(\boldsymbol{\Lambda^{-1}k}) = \Lambda\omega_A(\boldsymbol{k}). && \text{(B.13)}\end{aligned}$$

It can be shown that

$$\phi_A(\boldsymbol{\Lambda_1^{-1}k})\ \Lambda_2\psi^A(\boldsymbol{\Lambda_1^{-1}k}) = \Lambda_1\phi_A(\boldsymbol{k})\ \Lambda_1\Lambda_2\psi^A(\boldsymbol{k}). \tag{B.14}$$

$$\begin{aligned}\Lambda_1\phi_A(\boldsymbol{k})\ \Lambda_1\Lambda_2\psi^A(\boldsymbol{k}) &= {\Lambda_{1A}}^B\phi_B(\boldsymbol{\Lambda_1^{-1}k})\psi^C((\boldsymbol{\Lambda_1\Lambda_2})^{-1}\boldsymbol{k})(\Lambda_1\Lambda_2)^{-1}{}_C{}^A && \text{(B.15)}\\ &= {\Lambda_{1A}}^B(\Lambda_2)^{-1}{}_C{}^D(\Lambda_1)^{-1}{}_D{}^A\phi_B(\boldsymbol{\Lambda_1^{-1}k})\psi^C((\boldsymbol{\Lambda_1\Lambda_2})^{-1}\boldsymbol{k}) && \text{(B.16)}\\ &= (\Lambda_2)^{-1}{}_C{}^D\delta_D{}^B\phi_B(\boldsymbol{\Lambda_1^{-1}k})\psi^C((\boldsymbol{\Lambda_1\Lambda_2})^{-1}\boldsymbol{k}) && \text{(B.17)}\\ &= \phi_B(\boldsymbol{\Lambda_1^{-1}k})\psi^C(\boldsymbol{\Lambda_2^{-1}}(\boldsymbol{\Lambda_1^{-1}k}))(\Lambda_2)^{-1}{}_C{}^B && \text{(B.18)}\\ &= \phi_A(\boldsymbol{\Lambda_1^{-1}k})\ \Lambda_2\psi^A(\boldsymbol{\Lambda_1^{-1}k}). && \text{(B.19)}\end{aligned}$$

Furthermore, it can be shown

$$\begin{aligned} e^{i\Theta(\Lambda,\boldsymbol{k})}e^{i\Theta(\Lambda',\boldsymbol{\Lambda^{-1}k})} &= {\varepsilon_1}^B(\boldsymbol{k})\Lambda{\varepsilon_B}^1(\boldsymbol{k}){\varepsilon_1}^A(\boldsymbol{\Lambda^{-1}k})\Lambda'{\varepsilon_A}^1(\boldsymbol{\Lambda^{-1}k})\\ &= {\varepsilon_1}^B(\boldsymbol{k})\Lambda{\varepsilon_B}^1(\boldsymbol{k})\Lambda{\varepsilon_1}^A(\boldsymbol{k})\Lambda\Lambda'{\varepsilon_A}^1(\boldsymbol{k})\\ &= {\varepsilon_1}^B(\boldsymbol{k}){\varepsilon_B}^A\Lambda\Lambda'{\varepsilon_A}^1(\boldsymbol{k})\\ &= {\varepsilon_1}^A(\boldsymbol{k})\Lambda\Lambda'{\varepsilon_A}^1(\boldsymbol{k}) = e^{i\Theta(\Lambda\Lambda',\boldsymbol{k})}. && \text{(B.20)}\end{aligned}$$

This formula is used to show the composition law in section 5.7.

C Gauge transformation on the spin-frame level

Let us consider a transformation on the spin-frame level such that

$$\omega_A(\boldsymbol{k}) \quad \mapsto \quad \tilde{\omega}_A(\boldsymbol{k}) \;=\; \omega_A(\boldsymbol{k}) + \phi(\boldsymbol{k})\pi_A(\boldsymbol{k}). \tag{C.1}$$

Then the tetrads transform as

$$\begin{aligned}
k_a(\boldsymbol{k}) &\quad\mapsto\quad \tilde{k}_a(\boldsymbol{k}) \;=\; \pi_A(\boldsymbol{k})\pi_{A'}(\boldsymbol{k}) \;=\; k_a(\boldsymbol{k}), && \text{(C.2)}\\
\omega_a(\boldsymbol{k}) &\quad\mapsto\quad \tilde{\omega}_a(\boldsymbol{k}) \;=\; \tilde{\omega}_A(\boldsymbol{k})\tilde{\omega}_{A'}(\boldsymbol{k}) \;=\; \omega_a(\boldsymbol{k}) + \bar{\phi}(\boldsymbol{k})m_a(\boldsymbol{k}) + \phi(\boldsymbol{k})\bar{m}_a(\boldsymbol{k}) + |\phi(\boldsymbol{k})|^2 k_a(\boldsymbol{k}), && \text{(C.3)}\\
m_a(\boldsymbol{k}) &\quad\mapsto\quad \tilde{m}_a(\boldsymbol{k}) \;=\; \tilde{\omega}_A(\boldsymbol{k})\pi_{A'}(\boldsymbol{k}) \;=\; m_a(\boldsymbol{k}) + \phi(\boldsymbol{k})k_a(\boldsymbol{k}), && \text{(C.4)}\\
\bar{m}_a(\boldsymbol{k}) &\quad\mapsto\quad \tilde{\bar{m}}_a(\boldsymbol{k}) \;=\; \pi_A(\boldsymbol{k})\tilde{\omega}_{A'}(\boldsymbol{k}) \;=\; \bar{m}_a(\boldsymbol{k}) + \bar{\phi}(\boldsymbol{k})k_a(\boldsymbol{k}), && \text{(C.5)}\\
x_a(\boldsymbol{k}) &\quad\mapsto\quad \tilde{x}_a(\boldsymbol{k}) \;=\; x_a(\boldsymbol{k}) + \frac{1}{\sqrt{2}}\left(\phi(\boldsymbol{k}) + \bar{\phi}(\boldsymbol{k})\right)k_a(\boldsymbol{k}), && \text{(C.6)}\\
y_a(\boldsymbol{k}) &\quad\mapsto\quad \tilde{y}_a(\boldsymbol{k}) \;=\; y_a(\boldsymbol{k}) + \frac{i}{\sqrt{2}}\left(\phi(\boldsymbol{k}) - \bar{\phi}(\boldsymbol{k})\right)k_a(\boldsymbol{k}), && \text{(C.7)}\\
z_a(\boldsymbol{k}) &\quad\mapsto\quad \tilde{z}_a(\boldsymbol{k}) \;=\; z_a(\boldsymbol{k}) + \frac{1}{\sqrt{2}}\left(\bar{\phi}(\boldsymbol{k})m_a(\boldsymbol{k}) + \phi(\boldsymbol{k})\bar{m}_a(\boldsymbol{k}) + |\phi(\boldsymbol{k})|^2 k_a(\boldsymbol{k})\right), && \text{(C.8)}\\
t_a(\boldsymbol{k}) &\quad\mapsto\quad \tilde{t}_a(\boldsymbol{k}) \;=\; t_a(\boldsymbol{k}) + \frac{1}{\sqrt{2}}\left(\bar{\phi}(\boldsymbol{k})m_a(\boldsymbol{k}) + \phi(\boldsymbol{k})\bar{m}_a(\boldsymbol{k}) + |\phi(\boldsymbol{k})|^2 k_a(\boldsymbol{k})\right). && \text{(C.9)}
\end{aligned}$$

These calculations are done for formula (413) in section 5.11.

D Maxwell-corresponding vector space

All the calculus in this appendix is done for section 4.4

$$
\begin{aligned}
& \langle\Psi_{03}(1)|\,(a_0-a_3)_2\,|\Psi_{03}(1)\rangle \\
= & \sum_{n_0'=0,n_3'=0}\ \sum_{n_0=0,n_3=0} \langle n_0',n_3'|\bar{\Psi}(n_0',n_3')\,(a_0-a_3)_2\,\Psi(n_0,n_3)|n_0,n_3\rangle \\
= & \sum_{n_0'=0,n_3'=0}\ \sum_{n_0=0,n_3=0} \langle n_0',n_3'|\bar{\Psi}(n_0',n_3')\Psi(n_0,n_3)\sqrt{n_0+1}|n_0+1,n_3\rangle \\
- & \sum_{n_0'=0,n_3'=0}\ \sum_{n_0=0,n_3=1} \langle n_0',n_3'|\bar{\Psi}(n_0',n_3')\Psi(n_0,n_3)\sqrt{n_3}|n_0,n_3-1\rangle \\
= & \sum_{n_0'=0,n_3'=0}\ \sum_{n_0=0,n_3=0} \bar{\Psi}(n_0',n_3')\Psi(n_0,n_3)\sqrt{n_0+1}\delta_{n_0+1,n_0'}\delta_{n_3,n_3'} \\
- & \sum_{n_0'=0,n_3'=0}\ \sum_{n_0=0,n_3=0} \bar{\Psi}(n_0',n_3')\Psi(n_0,n_3+1)\sqrt{n_3+1}\delta_{n_0,n_0'}\delta_{n_3,n_3'} \\
= & \sum_{n_0=0,n_3=0} \left(\sqrt{n_0+1}\bar{\Psi}(n_0+1,n_3)\Psi(n_0,n_3)-\sqrt{n_3+1}\bar{\Psi}(n_0,n_3)\Psi(n_0,n_3+1)\right),
\end{aligned} \tag{D.1}
$$

$$
\begin{aligned}
& \langle\Psi_{03}(1)|\left(a_0^\dagger-a_3^\dagger\right)_2|\Psi_{03}(1)\rangle \\
= & \sum_{n_0'=0,n_3'=0}\ \sum_{n_0=0,n_3=0} \langle n_0',n_3'|\bar{\Psi}(n_0',n_3')\left(a_0^\dagger-a_3^\dagger\right)_2\Psi(n_0,n_3)|n_0,n_3\rangle \\
= & \sum_{n_0'=0,n_3'=0}\ \sum_{n_0=1,n_3=0} \langle n_0',n_3'|\bar{\Psi}(n_0',n_3')\Psi(n_0,n_3)\sqrt{n_0}|n_0-1,n_3\rangle \\
- & \sum_{n_0'=0,n_3'=0}\ \sum_{n_0=0,n_3=0} \langle n_0',n_3'|\bar{\Psi}(n_0',n_3')\Psi(n_0,n_3)\sqrt{n_3+1}|n_0,n_3+1\rangle \\
= & \sum_{n_0'=0,n_3'=0}\ \sum_{n_0=0,n_3=0} \bar{\Psi}(n_0',n_3')\Psi(n_0+1,n_3)\sqrt{n_0+1}\delta_{n_0,n_0'}\delta_{n_3,n_3'} \\
- & \sum_{n_0'=0,n_3'=0}\ \sum_{n_0=0,n_3=0} \bar{\Psi}(n_0',n_3')\Psi(n_0,n_3)\sqrt{n_3+1}\delta_{n_0,n_0'}\delta_{n_3+1,n_3'} \\
= & \sum_{n_0'=0,n_3'=0}\ \sum_{n_0=0,n_3=0} \left(\bar{\Psi}(n_0,n_3)\Psi(n_0+1,n_3)\sqrt{n_0+1}-\bar{\Psi}(n_0,n_3+1)\Psi(n_0,n_3)\sqrt{n_3+1}\right),
\end{aligned} \tag{D.2}
$$

$$
\begin{aligned}
& \langle \Psi_{03}(1)|\,(a_0 - a_3)^n\,|\Psi_{03}(1)\rangle \\
= \;& \sum_{k=0}^{n} (-)^k \left(\begin{array}{c} n \\ k \end{array} \right) \langle \Psi_{03}(1)| a_0^{n-k} a_3^k |\Psi_{03}(1)\rangle \\
= \;& \sum_{k=0}^{n} (-)^k \left(\begin{array}{c} n \\ k \end{array} \right) e^{-2} \\
\times \;& \sum_{n_0'=0, n_3'=0} \sum_{n_0=0, n_3=k} \langle n_0', n_3'| \frac{1}{\sqrt{n_0'! n_3'!}} \frac{1}{\sqrt{n_0! n_3!}} \sqrt{\frac{(n_0+n+k)!}{n_0!}} \sqrt{\frac{n_3!}{(n_3-k)!}} |n_3 - k, n_0 + n + k\rangle \\
= \;& \sum_{k=0}^{n} (-)^k \left(\begin{array}{c} n \\ k \end{array} \right) e^{-2} \\
\times \;& \sum_{n_0'=0, n_3'=0} \sum_{n_0=0, n_3=k} \frac{1}{\sqrt{n_0'! n_3'!}} \frac{1}{\sqrt{n_0! n_3!}} \sqrt{\frac{(n_0+n+k)!}{n_0!}} \sqrt{\frac{n_3!}{(n_3-k)!}} \delta_{n_0+n+k, n_0'} \delta_{n_3-k, n_3'} \\
= \;& \sum_{k=0}^{n} (-)^k \left(\begin{array}{c} n \\ k \end{array} \right) e^{-2} \\
\times \;& \sum_{n_0'=0, n_3'=0} \sum_{n_0=0, n_3=0} \frac{1}{\sqrt{n_0'! n_3'!}} \frac{1}{\sqrt{n_0! (n_3+k)!}} \sqrt{\frac{(n_0+n+k)!}{n_0!}} \sqrt{\frac{(n_3+k)!}{n_3!}} \delta_{n_0+n+k, n_0'} \delta_{n_3, n_3'} \\
= \;& \sum_{k=0}^{n} (-)^k \left(\begin{array}{c} n \\ k \end{array} \right) e^{-2} \sum_{n_0=0, n_3=0} \frac{1}{\sqrt{(n_0+n+k)! n_3!}} \frac{1}{\sqrt{n_0! (n_3+k)!}} \sqrt{\frac{(n_0+n+k)!}{n_0!}} \sqrt{\frac{(n_3+k)!}{n_3!}} \\
= \;& \sum_{k=0}^{n} (-)^k \left(\begin{array}{c} n \\ k \end{array} \right) e^{-2} \sum_{n_0=0, n_3=0} \frac{1}{n_0! n_3!} \\
= \;& \sum_{k=0}^{n} (-)^k \left(\begin{array}{c} n \\ k \end{array} \right) \;=\; \sum_{k=0}^{n} (-)^k \left(\begin{array}{c} n \\ k \end{array} \right) 1^{(n-k)} 1^k \;=\; (1-1)^n \;=\; 0,
\end{aligned}
\tag{D.3}
$$

$$
\begin{aligned}
& \langle\Psi_{03}(1)| \left(a_0^\dagger - a_3^\dagger\right)_2 (a_0 - a_3)_2 |\Psi_{03}(1)\rangle \\
= & \sum_{n_0'=0,n_3'=0} \sum_{n_0=0,n_3=0} \langle n_0', n_3'|\bar{\Psi}(n_0', n_3') \left(a_0^\dagger - a_3^\dagger\right)_2 (a_0 - a_3)_2 \Psi(n_0, n_3)|n_0, n_3\rangle \\
= & \sum_{n_0'=0,n_3'=0} \sum_{n_0=0,n_3=0} \langle n_0', n_3'|\bar{\Psi}(n_0', n_3')\Psi(n_0, n_3)\sqrt{n_0+1} \left(a_0^\dagger - a_3^\dagger\right)_2 |n_0+1, n_3\rangle \\
- & \sum_{n_0'=0,n_3'=0} \sum_{n_0=0,n_3=1} \langle n_0', n_3'|\bar{\Psi}(n_0', n_3')\Psi(n_0, n_3)\sqrt{n_3} \left(a_0^\dagger - a_3^\dagger\right)_2 |n_0, n_3-1\rangle \\
= & \sum_{n_0'=0,n_3'=0} \sum_{n_0=0,n_3=0} \langle n_0', n_3'|\bar{\Psi}(n_0', n_3')\Psi(n_0, n_3)\sqrt{n_0+1}\sqrt{n_0+1}|n_0, n_3\rangle \\
- & \sum_{n_0'=0,n_3'=0} \sum_{n_0=0,n_3=0} \langle n_0', n_3'|\bar{\Psi}(n_0', n_3')\Psi(n_0, n_3)\sqrt{n_0+1}\sqrt{n_3+1}|n_0+1, n_3+1\rangle \\
- & \sum_{n_0'=0,n_3'=0} \sum_{n_0=1,n_3=1} \langle n_0', n_3'|\bar{\Psi}(n_0', n_3')\Psi(n_0, n_3)\sqrt{n_3}\sqrt{n_0}|n_0-1, n_3-1\rangle \\
+ & \sum_{n_0'=0,n_3'=0} \sum_{n_0=0,n_3=1} \langle n_0', n_3'|\bar{\Psi}(n_0', n_3')\Psi(n_0, n_3)\sqrt{n_3}\sqrt{n_3}|n_0, n_3\rangle \\
= & \sum_{n_0'=0,n_3'=0} \sum_{n_0=0,n_3=0} \bar{\Psi}(n_0', n_3')\Psi(n_0, n_3)\sqrt{n_0+1}\sqrt{n_0+1}\delta_{n_0,n_0'}\delta_{n_3,n_3'} \\
- & \sum_{n_0'=0,n_3'=0} \sum_{n_0=0,n_3=0} \bar{\Psi}(n_0', n_3')\Psi(n_0, n_3)\sqrt{n_0+1}\sqrt{n_3+1}\delta_{n_0+1,n_0'}\delta_{n_3+1,n_3'} \\
- & \sum_{n_0'=0,n_3'=0} \sum_{n_0=0,n_3=0} \bar{\Psi}(n_0', n_3')\Psi(n_0+1, n_3+1)\sqrt{n_3+1}\sqrt{n_0+1}\delta_{n_0,n_0'}\delta_{n_3,n_3'} \\
+ & \sum_{n_0'=0,n_3'=0} \sum_{n_0=0,n_3=0} \bar{\Psi}(n_0', n_3')\Psi(n_0, n_3+1)\sqrt{n_3+1}\sqrt{n_3+1}\delta_{n_0,n_0'}\delta_{n_3+1,n_3'} \\
= & \sum_{n_0=0,n_3=0} \bar{\Psi}(n_0, n_3)\Psi(n_0, n_3)\sqrt{n_0+1}\sqrt{n_0+1} \\
- & \sum_{n_0=0,n_3=0} \bar{\Psi}(n_0+1, n_3+1)\Psi(n_0, n_3)\sqrt{n_0+1}\sqrt{n_3+1} \\
- & \sum_{n_0=0,n_3=0} \bar{\Psi}(n_0, n_3)\Psi(n_0+1, n_3+1)\sqrt{n_3+1}\sqrt{n_0+1} \\
+ & \sum_{n_0=0,n_3=0} \bar{\Psi}(n_0, n_3+1)\Psi(n_0, n_3+1)\sqrt{n_3+1}\sqrt{n_3+1} \\
= & \sum_{n_0=0,n_3=0} \frac{n_0+1}{n_0!n_3!} - \frac{\sqrt{n_0+1}\sqrt{n_3+1}}{\sqrt{(n_0+1)!(n_3+1)!}\sqrt{n_0!n_3!}} - \frac{\sqrt{n_0+1}\sqrt{n_3+1}}{\sqrt{(n_0+1)!(n_3+1)!}\sqrt{n_0!n_3!}} + \frac{n_3+1}{n_0!(n_3+1)!} \\
= & \sum_{n_0=0,n_3=0} \frac{n_0+1}{n_0!n_3!} - \frac{1}{n_0!n_3!} - \frac{1}{n_0!n_3!} + \frac{1}{n_0!n_3!} \\
= & \sum_{n_0=0,n_3=0} \frac{1}{n_0!n_3!} \\
= & \sum_{n_0=0,n_3=0} \frac{1}{n_0!n_3!} = 1,
\end{aligned}
\tag{D.4}
$$

$$
\begin{aligned}
& \langle\Psi_{03}(1)|\left(a_0-a_3\right)_2\left(a_0^\dagger-a_3^\dagger\right)_2|\Psi_{03}(1)\rangle \\
= & \sum_{n_0'=0,n_3'=0}\sum_{n_0=0,n_3=0}\langle n_0',n_3'|\bar{\Psi}(n_0',n_3')\left(a_0-a_3\right)_2\left(a_0^\dagger-a_3^\dagger\right)_2\Psi(n_0,n_3)|n_0,n_3\rangle \\
= & \sum_{n_0'=0,n_3'=0}\sum_{n_0=0,n_3=0}\langle n_0',n_3'|\bar{\Psi}(n_0',n_3')\left(a_0-a_3\right)_2\Psi(n_0,n_3)\sqrt{n_0}|n_0-1,n_3\rangle \\
- & \sum_{n_0'=0,n_3'=0}\sum_{n_0=0,n_3=0}\langle n_0',n_3'|\bar{\Psi}(n_0',n_3')\left(a_0-a_3\right)_2\Psi(n_0,n_3)\sqrt{n_3+1}|n_0,n_3+1\rangle \\
= & \sum_{n_0'=0,n_3'=0}\sum_{n_0=0,n_3=0}\langle n_0',n_3'|\bar{\Psi}(n_0',n_3')\Psi(n_0,n_3)\sqrt{n_0}\sqrt{n_0}|n_0,n_3\rangle \\
- & \sum_{n_0'=0,n_3'=0}\sum_{n_0=0,n_3=0}\langle n_0',n_3'|\bar{\Psi}(n_0',n_3')\Psi(n_0,n_3)\sqrt{n_0}\sqrt{n_3}|n_0-1,n_3-1\rangle \\
- & \sum_{n_0'=0,n_3'=0}\sum_{n_0=0,n_3=0}\langle n_0',n_3'|\bar{\Psi}(n_0',n_3')\Psi(n_0,n_3)\sqrt{n_3+1}\sqrt{n_0+1}|n_0+1,n_3+1\rangle \\
+ & \sum_{n_0'=0,n_3'=0}\sum_{n_0=0,n_3=0}\langle n_0',n_3'|\bar{\Psi}(n_0',n_3')\Psi(n_0,n_3)\sqrt{n_3+1}\sqrt{n_3+1}|n_0,n_3\rangle \\
= & \sum_{n_0'=0,n_3'=0}\sum_{n_0=0,n_3=0}\bar{\Psi}(n_0',n_3')\Psi(n_0,n_3)\sqrt{n_0}\sqrt{n_0}\delta_{n_0,n_0'}\delta_{n_3,n_3'} \\
- & \sum_{n_0'=0,n_3'=0}\sum_{n_0=0,n_3=0}\bar{\Psi}(n_0',n_3')\Psi(n_0,n_3)\sqrt{n_0}\sqrt{n_3}\delta_{n_0-1,n_0'}\delta_{n_3-1,n_3'} \\
- & \sum_{n_0'=0,n_3'=0}\sum_{n_0=0,n_3=0}\bar{\Psi}(n_0',n_3')\Psi(n_0,n_3)\sqrt{n_3+1}\sqrt{n_0+1}\delta_{n_0+1,n_0'}\delta_{n_3+1,n_3'} \\
+ & \sum_{n_0'=0,n_3'=0}\sum_{n_0=0,n_3=0}\bar{\Psi}(n_0',n_3')\Psi(n_0,n_3)\sqrt{n_3+1}\sqrt{n_3+1}\delta_{n_0,n_0'}\delta_{n_3,n_3'} \\
= & \sum_{n_0=0,n_3=0}\bar{\Psi}(n_0,n_3)\Psi(n_0,n_3)\sqrt{n_0}\sqrt{n_0} \\
- & \sum_{n_0=0,n_3=0}\bar{\Psi}(n_0-1,n_3-1)\Psi(n_0,n_3)\sqrt{n_0}\sqrt{n_3} \\
- & \sum_{n_0=0,n_3=0}\bar{\Psi}(n_0+1,n_3+1)\Psi(n_0,n_3)\sqrt{n_3+1}\sqrt{n_0+1} \\
+ & \sum_{n_0=0,n_3=0}\bar{\Psi}(n_0,n_3)\Psi(n_0,n_3)\sqrt{n_3+1}\sqrt{n_3+1} \\
= & \sum_{n_0=0,n_3=0}\left(\frac{n_0}{n_0!n_3!}-\frac{\sqrt{n_3}\sqrt{n_0}}{\sqrt{(n_0-1)!(n_3-1)!}\sqrt{n_0!n_3!}}-\frac{\sqrt{n_3+1}\sqrt{n_0+1}}{\sqrt{(n_0+1)!(n_3+1)!}\sqrt{n_0!n_3!}}+\frac{n_3+1}{n_0!n_3!}\right) \\
= & \sum_{n_0=0,n_3=0}\left(\frac{n_0}{n_0!n_3!}-\frac{n_3n_0}{n_0!n_3!}-\frac{1}{n_0!n_3!}+\frac{n_3+1}{n_0!n_3!}\right) \\
= & \sum_{n_0=0,n_3=0}\frac{1}{n_0!n_3!}=1. \qquad \text{(D.5)}
\end{aligned}
$$

E Four-vector algebra

The inner product of two vectors $A_{\boldsymbol{a}} = (A_0, \boldsymbol{A})$ and $B_{\boldsymbol{a}} = (B_0, \boldsymbol{B})$ is defined by

$$A \cdot B = A_0 B_0 - \boldsymbol{A} \cdot \boldsymbol{B} = A_0 B_0 - A_1 B_1 - A_2 B_2 - A_3 B_3. \tag{E.1}$$

This can be also written in terms of covariant and contravariant coordinates

$$A \cdot B \ = \ A_0 B^0 + A_1 B^1 + A_2 B^2 + A_3 B^3. \tag{E.2}$$

Lowering and raising of indexes is done by means of the metric tensor, which can be denoted in a matrix form

$$g^{\boldsymbol{ab}} \ = \ g_{\boldsymbol{ab}} \ = \ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{E.3}$$

so that

$$A^{\boldsymbol{a}} \ = \ g^{\boldsymbol{ab}} A_{\boldsymbol{b}} : \tag{E.4}$$

$$A^0 \ = \ A_0, \qquad A^{\boldsymbol{i}} = -A_{\boldsymbol{i}}. \tag{E.5}$$

By definition the first index is to be increased downwards from 0 to 3 (indicates the row number) and the second to increase to the right (indicates the kolumn number). It also follows that:

$$A \cdot B \ = \ A_{\boldsymbol{a}} B^{\boldsymbol{a}} \ = \ A^{\boldsymbol{a}} B_{\boldsymbol{a}}. \tag{E.6}$$

Having for an example matrix (316)

$$L_{\boldsymbol{a}}{}^{\boldsymbol{b}}(\Lambda, \boldsymbol{k}) \ = \ \begin{pmatrix} t_a(\boldsymbol{k})\Lambda\tilde{t}^a(\boldsymbol{k}) & -t_a(\boldsymbol{k})\Lambda\tilde{x}^a(\boldsymbol{k}) & -t_a(\boldsymbol{k})\Lambda\tilde{y}^a(\boldsymbol{k}) & -t_a(\boldsymbol{k})\Lambda\tilde{z}^a(\boldsymbol{k}) \\ x_a(\boldsymbol{k})\Lambda\tilde{t}^a(\boldsymbol{k}) & -x_a(\boldsymbol{k})\Lambda\tilde{x}^a(\boldsymbol{k}) & -x_a(\boldsymbol{k})\Lambda\tilde{y}^a(\boldsymbol{k}) & -x_a(\boldsymbol{k})\Lambda\tilde{z}^a(\boldsymbol{k}) \\ y_a(\boldsymbol{k})\Lambda\tilde{t}^a(\boldsymbol{k}) & -y_a(\boldsymbol{k})\Lambda\tilde{x}^a(\boldsymbol{k}) & -y_a(\boldsymbol{k})\Lambda\tilde{y}^a(\boldsymbol{k}) & -y_a(\boldsymbol{k})\Lambda\tilde{z}^a(\boldsymbol{k}) \\ z_a(\boldsymbol{k})\Lambda\tilde{t}^a(\boldsymbol{k}) & -z_a(\boldsymbol{k})\Lambda\tilde{x}^a(\boldsymbol{k}) & -z_a(\boldsymbol{k})\Lambda\tilde{y}^a(\boldsymbol{k}) & -z_a(\boldsymbol{k})\Lambda\tilde{z}^a(\boldsymbol{k}) \end{pmatrix} \tag{E.7}$$

$$= \ \begin{pmatrix} 1 + \frac{|\phi|^2}{2} & -|\phi|\cos(\xi + 2\Theta) & |\phi|\sin(\xi + 2\Theta) & -\frac{|\phi|^2}{2} \\ -|\phi|\cos\xi & \cos 2\Theta & -\sin 2\Theta & |\phi|\cos\xi \\ |\phi|\sin\xi & \sin 2\Theta & \cos 2\Theta & -|\phi|\sin\xi \\ \frac{|\phi|^2}{2} & -|\phi|\cos(\xi + 2\Theta) & |\phi|\sin(\xi + 2\Theta) & 1 - \frac{|\phi|^2}{2} \end{pmatrix}, \tag{E.8}$$

we lower and raise the indices $\boldsymbol{ab}$ also by means of the metric tensor. This means that to lower the second index $\boldsymbol{b}$ columns from 1 to 3 have to be multiplied by -1, i.e. $L_{\boldsymbol{ab}} = L_{\boldsymbol{a}}{}^{\boldsymbol{c}} g_{\boldsymbol{cb}}$

$$L_{\boldsymbol{ab}}(\Lambda, \boldsymbol{k}) \ = \ \begin{pmatrix} t_a(\boldsymbol{k})\Lambda\tilde{t}^a(\boldsymbol{k}) & t_a(\boldsymbol{k})\Lambda\tilde{x}^a(\boldsymbol{k}) & t_a(\boldsymbol{k})\Lambda\tilde{y}^a(\boldsymbol{k}) & t_a(\boldsymbol{k})\Lambda\tilde{z}^a(\boldsymbol{k}) \\ x_a(\boldsymbol{k})\Lambda\tilde{t}^a(\boldsymbol{k}) & x_a(\boldsymbol{k})\Lambda\tilde{x}^a(\boldsymbol{k}) & x_a(\boldsymbol{k})\Lambda\tilde{y}^a(\boldsymbol{k}) & x_a(\boldsymbol{k})\Lambda\tilde{z}^a(\boldsymbol{k}) \\ y_a(\boldsymbol{k})\Lambda\tilde{t}^a(\boldsymbol{k}) & y_a(\boldsymbol{k})\Lambda\tilde{x}^a(\boldsymbol{k}) & y_a(\boldsymbol{k})\Lambda\tilde{y}^a(\boldsymbol{k}) & y_a(\boldsymbol{k})\Lambda\tilde{z}^a(\boldsymbol{k}) \\ z_a(\boldsymbol{k})\Lambda\tilde{t}^a(\boldsymbol{k}) & z_a(\boldsymbol{k})\Lambda\tilde{x}^a(\boldsymbol{k}) & z_a(\boldsymbol{k})\Lambda\tilde{y}^a(\boldsymbol{k}) & z_a(\boldsymbol{k})\Lambda\tilde{z}^a(\boldsymbol{k}) \end{pmatrix} \tag{E.9}$$

$$= \ \begin{pmatrix} 1 + \frac{|\phi|^2}{2} & |\phi|\cos(\xi + 2\Theta) & -|\phi|\sin(\xi + 2\Theta) & \frac{|\phi|^2}{2} \\ -|\phi|\cos\xi & -\cos 2\Theta & \sin 2\Theta & -|\phi|\cos\xi \\ |\phi|\sin\xi & -\sin 2\Theta & -\cos 2\Theta & |\phi|\sin\xi \\ \frac{|\phi|^2}{2} & |\phi|\cos(\xi + 2\Theta) & -|\phi|\sin(\xi + 2\Theta) & -1 + \frac{|\phi|^2}{2} \end{pmatrix}. \tag{E.10}$$

To higher the first index $\boldsymbol{a}$ rows from 1 to 3 in (316) have to be multiplied by -1, i.e. $L^{\boldsymbol{ab}} = g^{\boldsymbol{cb}} L_{\boldsymbol{c}}{}^{\boldsymbol{b}}$

$$L^{\boldsymbol{ab}}(\Lambda, \boldsymbol{k}) \ = \ \begin{pmatrix} t_a(\boldsymbol{k})\Lambda\tilde{t}^a(\boldsymbol{k}) & -t_a(\boldsymbol{k})\Lambda\tilde{x}^a(\boldsymbol{k}) & -t_a(\boldsymbol{k})\Lambda\tilde{y}^a(\boldsymbol{k}) & -t_a(\boldsymbol{k})\Lambda\tilde{z}^a(\boldsymbol{k}) \\ -x_a(\boldsymbol{k})\Lambda\tilde{t}^a(\boldsymbol{k}) & x_a(\boldsymbol{k})\Lambda\tilde{x}^a(\boldsymbol{k}) & x_a(\boldsymbol{k})\Lambda\tilde{y}^a(\boldsymbol{k}) & x_a(\boldsymbol{k})\Lambda\tilde{z}^a(\boldsymbol{k}) \\ -y_a(\boldsymbol{k})\Lambda\tilde{t}^a(\boldsymbol{k}) & y_a(\boldsymbol{k})\Lambda\tilde{x}^a(\boldsymbol{k}) & y_a(\boldsymbol{k})\Lambda\tilde{y}^a(\boldsymbol{k}) & y_a(\boldsymbol{k})\Lambda\tilde{z}^a(\boldsymbol{k}) \\ -z_a(\boldsymbol{k})\Lambda\tilde{t}^a(\boldsymbol{k}) & z_a(\boldsymbol{k})\Lambda\tilde{x}^a(\boldsymbol{k}) & z_a(\boldsymbol{k})\Lambda\tilde{y}^a(\boldsymbol{k}) & z_a(\boldsymbol{k})\Lambda\tilde{z}^a(\boldsymbol{k}) \end{pmatrix} \tag{E.11}$$

$$= \ \begin{pmatrix} 1 + \frac{|\phi|^2}{2} & -|\phi|\cos(\xi + 2\Theta) & |\phi|\sin(\xi + 2\Theta) & -\frac{|\phi|^2}{2} \\ |\phi|\cos\xi & -\cos 2\Theta & \sin 2\Theta & -|\phi|\cos\xi \\ -|\phi|\sin\xi & -\sin 2\Theta & -\cos 2\Theta & |\phi|\sin\xi \\ -\frac{|\phi|^2}{2} & |\phi|\cos(\xi + 2\Theta) & -|\phi|\sin(\xi + 2\Theta) & -1 + \frac{|\phi|^2}{2} \end{pmatrix}, \tag{E.12}$$

Finally to raise the first index $\boldsymbol{a}$ and higher the second index $\boldsymbol{b}$ columns from 1 to 3 in (316) have to be multiplied by -1 and rows form 1 to 3 also multiplied by -1, i.e. $L^{\boldsymbol{a}}{}_{\boldsymbol{b}} = g^{\boldsymbol{ac}} L_{\boldsymbol{c}}{}^{\boldsymbol{d}} g_{\boldsymbol{db}}$

$$L^{\boldsymbol{a}}{}_{\boldsymbol{b}}(\Lambda, \boldsymbol{k}) = \begin{pmatrix} t_a(\boldsymbol{k})\Lambda\tilde{t}^a(\boldsymbol{k}) & t_a(\boldsymbol{k})\Lambda\tilde{x}^a(\boldsymbol{k}) & t_a(\boldsymbol{k})\Lambda\tilde{y}^a(\boldsymbol{k}) & t_a(\boldsymbol{k})\Lambda\tilde{z}^a(\boldsymbol{k}) \\ -x_a(\boldsymbol{k})\Lambda\tilde{t}^a(\boldsymbol{k}) & -x_a(\boldsymbol{k})\Lambda\tilde{x}^a(\boldsymbol{k}) & -x_a(\boldsymbol{k})\Lambda\tilde{y}^a(\boldsymbol{k}) & -x_a(\boldsymbol{k})\Lambda\tilde{z}^a(\boldsymbol{k}) \\ -y_a(\boldsymbol{k})\Lambda\tilde{t}^a(\boldsymbol{k}) & -y_a(\boldsymbol{k})\Lambda\tilde{x}^a(\boldsymbol{k}) & -y_a(\boldsymbol{k})\Lambda\tilde{y}^a(\boldsymbol{k}) & -y_a(\boldsymbol{k})\Lambda\tilde{z}^a(\boldsymbol{k}) \\ -z_a(\boldsymbol{k})\Lambda\tilde{t}^a(\boldsymbol{k}) & -z_a(\boldsymbol{k})\Lambda\tilde{x}^a(\boldsymbol{k}) & -z_a(\boldsymbol{k})\Lambda\tilde{y}^a(\boldsymbol{k}) & -z_a(\boldsymbol{k})\Lambda\tilde{z}^a(\boldsymbol{k}) \end{pmatrix} \tag{E.13}$$

$$= \begin{pmatrix} 1 + \frac{|\phi|^2}{2} & |\phi|\cos(\xi + 2\Theta) & -|\phi|\sin(\xi + 2\Theta) & \frac{|\phi|^2}{2} \\ |\phi|\cos\xi & \cos 2\Theta & -\sin 2\Theta & |\phi|\cos\xi \\ -|\phi|\sin\xi & \sin 2\Theta & \cos 2\Theta & -|\phi|\sin\xi \\ -\frac{|\phi|^2}{2} & -|\phi|\cos(\xi + 2\Theta) & |\phi|\sin(\xi + 2\Theta) & 1 - \frac{|\phi|^2}{2} \end{pmatrix}. \tag{E.14}$$

F Unitary matrix expansions of creation and annihilation operators

Let us first give a definition of an unitary matrix

$$\boldsymbol{U}^\dagger \boldsymbol{U} \;=\; \boldsymbol{U}\boldsymbol{U}^\dagger \;=\; 1. \tag{F.1}$$

The matrix exponentials are defined as

$$\exp(\boldsymbol{A}) \;=\; \sum_{n=0}^{\infty} \frac{\boldsymbol{A}^n}{n!}. \tag{F.2}$$

Now any unitary matrix $\boldsymbol{U}$ can be written as the exponential of an anti-Hermitian matrix

$$\boldsymbol{U} \;=\; \exp(\boldsymbol{A}), \qquad \boldsymbol{A}^\dagger \;=\; -\boldsymbol{A}, \tag{F.3}$$

then it follows that

$$\exp(\boldsymbol{A})^\dagger \exp(\boldsymbol{A}) \;=\; \exp(-\boldsymbol{A})\exp(\boldsymbol{A}) \;=\; \mathbf{1}. \tag{F.4}$$

Furthermore, the following are true

$$\exp(-\boldsymbol{A})\exp(\boldsymbol{A}) \;=\; \mathbf{1}, \tag{F.5}$$

$$\exp(\boldsymbol{A})^\dagger \;=\; \exp(\boldsymbol{A}^\dagger), \tag{F.6}$$

$$\boldsymbol{B}\exp(\boldsymbol{A})\boldsymbol{B}^{-1} \;=\; \exp(\boldsymbol{B}\boldsymbol{A}\boldsymbol{B}^{-1}), \tag{F.7}$$

$$\exp(-\boldsymbol{A})\boldsymbol{B}\exp(\boldsymbol{A}) \;=\; \boldsymbol{B} + [\boldsymbol{B},\boldsymbol{A}] + \frac{1}{2!}[[\boldsymbol{B},\boldsymbol{A}],\boldsymbol{A}] + \frac{1}{3!}[[[\boldsymbol{B},\boldsymbol{A}],\boldsymbol{A}],\boldsymbol{A}] + \ldots \tag{F.8}$$

Also when $\boldsymbol{A}$ and $\boldsymbol{B}$ are two non-commuting operators and

$$[\boldsymbol{A},[\boldsymbol{A},\boldsymbol{B}]] \;=\; [\boldsymbol{B},[\boldsymbol{B},\boldsymbol{A}]] \;=\; 0, \tag{F.9}$$

we have

$$\exp(\boldsymbol{A})\exp(\boldsymbol{B}) \;=\; \exp(\boldsymbol{A}+\boldsymbol{B})\exp\left(\frac{1}{2}[\boldsymbol{A},\boldsymbol{B}]\right), \tag{F.10}$$

$$\exp(\boldsymbol{A})\exp(\boldsymbol{B}) \;=\; \exp(\boldsymbol{B})\exp(\boldsymbol{A})\exp\left([\boldsymbol{A},\boldsymbol{B}]\right). \tag{F.11}$$

Moreover, for a non-central element $[\boldsymbol{A},\boldsymbol{B}]$ we have

$$\exp(\boldsymbol{A})\exp(\boldsymbol{B}) \;=\; \exp\left(\boldsymbol{B} + [\boldsymbol{A},\boldsymbol{B}] + \frac{1}{2!}[\boldsymbol{A},[\boldsymbol{A},\boldsymbol{B}]] + \frac{1}{3!}[\boldsymbol{A},[\boldsymbol{A},[\boldsymbol{A},\boldsymbol{B}]]] + \ldots\right)\exp(\boldsymbol{A}). \tag{F.12}$$

The last formula is knows as the Baker-Campbell-Hausdorff (BCH) expansion.

Now let us consider an unitary transformation on annihilation operators:

$$\tilde{a}_k \;=\; \exp(\hat{\theta})^\dagger a_k \exp(\hat{\theta}) \;=\; \exp(-\hat{\theta}) a_k \exp(\hat{\theta}) \tag{F.13}$$

where $\hat{\theta}$ is expressed it terms of creation and annihilation operators:

$$[a_i, a_j^\dagger] \;=\; \delta_{ij}, \qquad i,j \;=\; 1,2,3, \tag{F.14}$$

$$\hat{\theta} \;=\; \sum_{ij} \theta_{ij},\; a_i^\dagger a_j \qquad i,j \;=\; 1,2,3. \tag{F.15}$$

From the unitarity of this transformation one gets conditions on the θ_{ij} parameters:

$$\hat{\theta}^\dagger \;=\; \sum_{ij} \theta_{ij}^*\; a_j^\dagger a_i \;=\; \sum_{ij} \theta_{ji}^*\; a_i^\dagger a_j \;=\; -\,\hat{\theta} \;=\; -\sum_{ij} \theta_{ij}\; a_i^\dagger a_j \;\;\Rightarrow\;\; \theta_{ji}^* \;=\; -\,\theta_{ij}. \tag{F.16}$$

From the BCH formula (F.8) we get the following unitary matrix expansions

$$\tilde{a}_k = \exp\left(\sum_{ij}\theta_{ij}\ a_i^\dagger a_j\right) a_k \exp\left(-\sum_{ij}\theta_{ij}\ a_i^\dagger a_j\right) = \sum_l \exp\left(-\boldsymbol{\theta}\right)_{kl}\ a_l, \tag{F.17}$$

$$\tilde{a}_k = \exp\left(-\sum_{ij}\theta_{ij}\ a_i^\dagger a_j\right) a_k \exp\left(\sum_{ij}\theta_{ij}\ a_i^\dagger a_j\right) = \sum_l \exp\left(\boldsymbol{\theta}\right)_{kl}\ a_l, \tag{F.18}$$

$$\tilde{a}_k^\dagger = \exp\left(\sum_{ij}\theta_{ij}\ a_i^\dagger a_j\right) a_k^\dagger \exp\left(-\sum_{ij}\theta_{ij}\ a_i^\dagger a_j\right) = \sum_l a_l^\dagger \exp\left(\boldsymbol{\theta}\right)_{lk}, \tag{F.19}$$

$$\tilde{a}_k^\dagger = \exp\left(-\sum_{ij}\theta_{ij}\ a_i^\dagger a_j\right) a_k^\dagger \exp\left(\sum_{ij}\theta_{ij}\ a_i^\dagger a_j\right) = \sum_l a_l^\dagger \exp\left(-\boldsymbol{\theta}\right)_{lk}. \tag{F.20}$$

Now let us express $\hat{\theta}$ in terms of four-dimensional creation and annihilation operators such that:

$$[a_{\boldsymbol{a}}, a_{\boldsymbol{b}}^\dagger] = -\,g_{\boldsymbol{ab}}, \qquad \boldsymbol{a}, \boldsymbol{b} = 0, 1, 2, 3, \tag{F.21}$$

in the form

$$\hat{\theta} = \theta^{\boldsymbol{ab}} a_{\boldsymbol{a}}^\dagger a_{\boldsymbol{b}}. \tag{F.22}$$

From the unitarity of this transformation one gets conditions on the $\theta^{\boldsymbol{ab}}$ parameters:

$$\hat{\theta}^\dagger = \theta^{*\boldsymbol{ab}} a_{\boldsymbol{b}}^\dagger a_{\boldsymbol{a}} = \theta^{*\boldsymbol{ba}} a_{\boldsymbol{a}}^\dagger a_{\boldsymbol{b}} = -\,\hat{\theta} = -\,\theta^{\boldsymbol{ab}} a_{\boldsymbol{a}}^\dagger a_{\boldsymbol{b}} \;\Rightarrow\; \theta^{*\boldsymbol{ba}} = -\,\theta^{\boldsymbol{ab}}. \tag{F.23}$$

From the BCH formula we get the unitary matrix expansions for four-dimensional ladder operators

$$\tilde{a}_{\boldsymbol{c}} = \exp(\theta^{\boldsymbol{b}}{}_{\boldsymbol{a}} a^{\boldsymbol{a}\dagger} a_{\boldsymbol{b}}) a_{\boldsymbol{c}} \exp(-\theta^{\boldsymbol{b}}{}_{\boldsymbol{a}} a^{\boldsymbol{a}\dagger} a_{\boldsymbol{b}}) = \exp(-\boldsymbol{\theta})^{\boldsymbol{d}}{}_{\boldsymbol{c}}\ a_{\boldsymbol{d}}, \tag{F.24}$$

$$\tilde{a}_{\boldsymbol{c}} = \exp(-\theta^{\boldsymbol{b}}{}_{\boldsymbol{a}} a^{\boldsymbol{a}\dagger} a_{\boldsymbol{b}}) a_{\boldsymbol{c}} \exp(\theta^{\boldsymbol{b}}{}_{\boldsymbol{a}} a^{\boldsymbol{a}\dagger} a_{\boldsymbol{b}}) = \exp(\boldsymbol{\theta})^{\boldsymbol{d}}{}_{\boldsymbol{c}}\ a_{\boldsymbol{d}}, \tag{F.25}$$

$$\tilde{a}_{\boldsymbol{c}}^\dagger = \exp(\theta^{\boldsymbol{b}}{}_{\boldsymbol{a}} a^{\boldsymbol{a}\dagger} a_{\boldsymbol{b}}) a_{\boldsymbol{c}}^\dagger \exp(-\theta^{\boldsymbol{b}}{}_{\boldsymbol{a}} a^{\boldsymbol{a}\dagger} a_{\boldsymbol{b}}) = \exp(\boldsymbol{\theta})_{\boldsymbol{c}}{}^{\boldsymbol{d}} a_{\boldsymbol{d}}^\dagger, \tag{F.26}$$

$$\tilde{a}_{\boldsymbol{c}}^\dagger = \exp(-\theta^{\boldsymbol{b}}{}_{\boldsymbol{a}} a^{\boldsymbol{a}\dagger} a_{\boldsymbol{b}}) a_{\boldsymbol{c}}^\dagger \exp(\theta^{\boldsymbol{b}}{}_{\boldsymbol{a}} a^{\boldsymbol{a}\dagger} a_{\boldsymbol{b}}) = \exp(-\boldsymbol{\theta})_{\boldsymbol{c}}{}^{\boldsymbol{d}} a_{\boldsymbol{d}}^\dagger. \tag{F.27}$$

F.1 Linear polarizations

Any linear polarization for N-oscillator representation can be defined due to a transformation on $a_1(\boldsymbol{k}, N)$ and $a_2(\boldsymbol{k}, N)$ annihilation operators which correspond to linear polarizations in x and y direction:

$$\begin{aligned} L_\theta(N)^\dagger a_1(\boldsymbol{k}, N) L_\theta(N) &= \cos\theta(\boldsymbol{k}) a_1(\boldsymbol{k}, N) + \sin\theta(\boldsymbol{k}) a_2(\boldsymbol{k}, N) = a_\theta(\boldsymbol{k}, N), \\ L_\theta(N)^\dagger a_2(\boldsymbol{k}, N) L_\theta(N) &= -\sin\theta(\boldsymbol{k}) a_1(\boldsymbol{k}, N) + \cos\theta(\boldsymbol{k}) a_2(\boldsymbol{k}, N) = a_{\theta'}(\boldsymbol{k}, N), \end{aligned} \tag{F.28}$$

where

$$L_\theta(N) = L_\theta(1)^{\otimes N}, \qquad L_\theta(1) = \int d\Gamma(\boldsymbol{k}) |\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes \exp\Big(\theta(\boldsymbol{k})(a_1^\dagger a_2 - a_2^\dagger a_1)\Big). \tag{F.29}$$

Here $a_\theta(\boldsymbol{k}, N)$ and $a_{\theta'}(\boldsymbol{k}, N)$ hold the same commutation relation as $a_1(\boldsymbol{k}, N)$ and $a_2(\boldsymbol{k}, N)$

$$[a_\theta(\boldsymbol{k}, N), a_\theta(\boldsymbol{k}', N)^\dagger] = I(\boldsymbol{k}, N)\delta_\Gamma(\boldsymbol{k}, \boldsymbol{k}') \tag{F.30}$$

and $\theta'(\boldsymbol{k}) = \theta(\boldsymbol{k}) + \frac{\pi}{2}$. One may write this transformation in matrix form:

$$\begin{pmatrix} a_\theta(\boldsymbol{k}, N) \\ a_{\theta'}(\boldsymbol{k}, N) \end{pmatrix} = \begin{pmatrix} \cos\theta(\boldsymbol{k}) & \sin\theta(\boldsymbol{k}) \\ \cos\theta'(\boldsymbol{k}) & \sin\theta'(\boldsymbol{k}) \end{pmatrix} \begin{pmatrix} a_1(\boldsymbol{k}, N) \\ a_2(\boldsymbol{k}, N) \end{pmatrix} \tag{F.31}$$

$$= \begin{pmatrix} \cos\theta(\boldsymbol{k}) & \sin\theta(\boldsymbol{k}) \\ -\sin\theta(\boldsymbol{k}) & \cos\theta(\boldsymbol{k}) \end{pmatrix} \begin{pmatrix} a_1(\boldsymbol{k}, N) \\ a_2(\boldsymbol{k}, N) \end{pmatrix}. \tag{F.32}$$

One may also find an inverse relation

$$\begin{aligned} a_1(\boldsymbol{k},N) &= \cos\theta(\boldsymbol{k})a_\theta(\boldsymbol{k},N) - \sin\theta(\boldsymbol{k})a_{\theta'}(\boldsymbol{k},N), \\ a_2(\boldsymbol{k},N) &= \sin\theta(\boldsymbol{k})a_\theta(\boldsymbol{k},N) + \cos\theta(\boldsymbol{k})a_{\theta'}(\boldsymbol{k},N). \end{aligned} \tag{F.33}$$

It should be stressed that, in the case of reducible quantization, $\theta(\boldsymbol{k})$ is a function of $\boldsymbol{k}$ and not just a parameter. This was discussed further in chapter 7, where it turns out that the dependence on momentum is significant for relativistic background.

F.2 Circular polarizations

Circular polarizations can be defined by a transformation

$$\begin{aligned} C_\theta(N)^\dagger a_1(\boldsymbol{k},N)C_\theta(N) &= \frac{1}{\sqrt{2}}\left(a_1(\boldsymbol{k},N)+ia_2(\boldsymbol{k},N)\right) = a_-(\boldsymbol{k},N), \\ C_\theta(N)^\dagger a_2(\boldsymbol{k},N)C_\theta(N) &= \frac{1}{\sqrt{2}}\left(a_1(\boldsymbol{k},N)-ia_2(\boldsymbol{k},N)\right) = a_+(\boldsymbol{k},N), \end{aligned} \tag{F.34}$$

where

$$C_{\theta_{ij}}(N) = C_{\theta_{ij}}(1)^{\otimes N}, \qquad C_{\theta_{ij}}(1) = \int d\Gamma(\boldsymbol{k})|\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes \exp\left(\sum_{ij=1,2}\theta_{ij}\ a_i^\dagger a_j\right). \tag{F.35}$$

Here θ_{ij} are coefficients of this transformation:

$$\begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}\pi}{9}i - \frac{\pi}{4}i & -\frac{\sqrt{3}\pi}{9} + \frac{\sqrt{3}\pi}{9}i \\ \frac{\sqrt{3}\pi}{9} + \frac{\sqrt{3}\pi}{9}i & -\frac{\sqrt{3}\pi}{9}i - \frac{\pi}{4}i \end{pmatrix}, \tag{F.36}$$

$$\begin{pmatrix} a_-(\boldsymbol{k},N) \\ a_+(\boldsymbol{k},N) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & +i \\ 1 & -i \end{pmatrix}\begin{pmatrix} a_1(\boldsymbol{k},N) \\ a_2(\boldsymbol{k},N) \end{pmatrix}, \tag{F.37}$$

where $a_\pm(\boldsymbol{k},N)$ hold the commutation relation

$$[a_\pm(\boldsymbol{k},N), a_\pm(\boldsymbol{k}',N)^\dagger] = I(\boldsymbol{k},N)\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}'). \tag{F.38}$$

The $_+$ index stands for right-handed polarization and $_-$ for left-handed ones. The relation between circular polarizations and any linear polarizations is

$$\begin{aligned} \begin{pmatrix} a_-(\boldsymbol{k},N) \\ a_+(\boldsymbol{k},N) \end{pmatrix} &= \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\theta(\boldsymbol{k})} & e^{i\theta'(\boldsymbol{k})} \\ e^{-i\theta(\boldsymbol{k})} & e^{-i\theta'(\boldsymbol{k})} \end{pmatrix}\begin{pmatrix} a_\theta(\boldsymbol{k},N) \\ a_{\theta'}(\boldsymbol{k},N) \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\theta(\boldsymbol{k})} & ie^{i\theta(\boldsymbol{k})} \\ e^{-i\theta(\boldsymbol{k})} & -ie^{-i\theta(\boldsymbol{k})} \end{pmatrix}\begin{pmatrix} a_\theta(\boldsymbol{k},N) \\ a_{\theta'}(\boldsymbol{k},N) \end{pmatrix}. \end{aligned} \tag{F.39}$$

Then the correspondences between the ladder operators in circular basis and in linear basis can be written as

$$a_s(\boldsymbol{k},N) = \frac{1}{\sqrt{2}}e^{-is\theta(\boldsymbol{k})}\left(a_\theta(\boldsymbol{k},N) - isa_{\theta'}(\boldsymbol{k},N)\right), \tag{F.40}$$

$$a_\theta(\boldsymbol{k},N) = \frac{1}{\sqrt{2}}\sum_{s=\pm}a_s(\boldsymbol{k},N)e^{is\theta(\boldsymbol{k})}. \tag{F.41}$$

G Displacement operator calculations

All calculus form this appendix are used for section 4.7. Let us start from $N = 1$ representation. In this case the displacement operator is defined as

$$\mathcal{D}(\alpha,1) = \exp\left(\int d\Gamma(\boldsymbol{k})\Big(\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}a_{\boldsymbol{a}}(\boldsymbol{k},1) - \text{H.c.}\Big)\right) \tag{G.1}$$

Acting with operator (G.1) on ground state we get an coherent state

$$\mathcal{D}(\alpha,1)|O(1)\rangle = |\alpha(1)\rangle. \tag{G.2}$$

Coherent states can be expressed by

$$\begin{aligned}
|\alpha(1)\rangle &= \mathcal{D}(\alpha,1)|O(1)\rangle \\
&= \int d\Gamma(\boldsymbol{k})|\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes \exp\Big(\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}a_{\boldsymbol{a}} - \text{H.c.}\Big)|O(1)\rangle \\
&= \int d\Gamma(\boldsymbol{k})|\boldsymbol{k}\rangle\langle\boldsymbol{k}| \otimes \exp\Big(-\alpha^1(\boldsymbol{k})a_1^\dagger\Big)\exp\left(\frac{1}{2}\overline{\alpha^1(\boldsymbol{k})}\alpha_1(\boldsymbol{k})\right)\exp\Big(-\alpha^2(\boldsymbol{k})a_2^\dagger\Big)\exp\left(\frac{1}{2}\overline{\alpha^2(\boldsymbol{k})}\alpha_2(\boldsymbol{k})\right) \\
&\times \exp\Big(-\alpha^3(\boldsymbol{k})a_3^\dagger\Big)\exp\left(\frac{1}{2}\overline{\alpha^3(\boldsymbol{k})}\alpha_3(\boldsymbol{k})\right)\exp\Big(\overline{\alpha^0(\boldsymbol{k})}a_0\Big)\exp\left(-\frac{1}{2}\overline{\alpha^0(\boldsymbol{k})}\alpha_0(\boldsymbol{k})\right)|O(1)\rangle \\
&= \int d\Gamma(\boldsymbol{k})|\boldsymbol{k}\rangle\langle\boldsymbol{k}|\exp\left(-\frac{1}{2}\left(|\alpha_1(\boldsymbol{k})|^2 + |\alpha_2(\boldsymbol{k})|^2 + |\alpha_3(\boldsymbol{k})|^2 + |\alpha_0(\boldsymbol{k})|^2\right)\right) \\
&\otimes \exp\Big(\alpha_1(\boldsymbol{k})a_1^\dagger\Big)\exp\Big(\alpha_2(\boldsymbol{k})a_2^\dagger\Big)\exp\Big(\alpha_3(\boldsymbol{k})a_3^\dagger\Big)\exp\Big(\overline{\alpha_0(\boldsymbol{k})}a_0\Big)\int d\Gamma(\boldsymbol{k}')O(\boldsymbol{k}')|\boldsymbol{k}',0,0,0,0\rangle \\
&= \int d\Gamma(\boldsymbol{k})O(\boldsymbol{k})\exp\left(-\frac{1}{2}\left(|\alpha_1(\boldsymbol{k})|^2 + |\alpha_2(\boldsymbol{k})|^2 + |\alpha_3(\boldsymbol{k})|^2 + |\alpha_0(\boldsymbol{k})|^2\right)\right) \\
&\quad \sum_{n_1,n_2,n_3,n_0}^{\infty}\frac{(\alpha_1(\boldsymbol{k}))^{n_1}(\alpha_2(\boldsymbol{k}))^{n_2}(\alpha_3(\boldsymbol{k}))^{n_3}(\overline{\alpha_0(\boldsymbol{k})})^{n_0}}{\sqrt{n_1!n_2!n_3!n_0!}}|\boldsymbol{k},n_1,n_2,n_3,n_0\rangle.
\end{aligned} \tag{G.3}$$

For the N representation we can write

$$\begin{aligned}
\mathcal{D}(\alpha,N) &= \exp\left(\int d\Gamma(\boldsymbol{k})\Big(\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}\frac{1}{\sqrt{N}}\sum_{n=1}^{N}a_{\boldsymbol{a}}(\boldsymbol{k},1)^{(n)}\Big)\right)\exp\left(-\int d\Gamma(\boldsymbol{k})\Big(\alpha^{\boldsymbol{a}}(\boldsymbol{k})\frac{1}{\sqrt{N}}\sum_{n=1}^{N}\left(a_{\boldsymbol{a}}(\boldsymbol{k},1)^\dagger\right)^{(n)}\Big)\right) \\
&\times \exp\left(\frac{1}{2}\int d\Gamma(\boldsymbol{k})\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}\alpha_{\boldsymbol{a}}(\boldsymbol{k})\frac{1}{N}\sum_{n=1}^{N}I(\boldsymbol{k},1)^{(n)}\right) \\
&= \prod_{n=1}^{N}\exp\left(\int d\Gamma(\boldsymbol{k})\Big(\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}\frac{1}{\sqrt{N}}a_{\boldsymbol{a}}(\boldsymbol{k},1)^{(n)}\Big)\right)\prod_{n=1}^{N}\exp\left(-\int d\Gamma(\boldsymbol{k})\Big(\alpha^{\boldsymbol{a}}(\boldsymbol{k})\frac{1}{\sqrt{N}}\left(a_{\boldsymbol{a}}(\boldsymbol{k},1)^\dagger\right)^{(n)}\Big)\right) \\
&\times \prod_{n=1}^{N}\exp\left(\frac{1}{2}\int d\Gamma(\boldsymbol{k})\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}\alpha_{\boldsymbol{a}}(\boldsymbol{k})\frac{1}{N}I(\boldsymbol{k},1)^{(n)}\right) &(\text{G.4}) \\
&= \exp\left(\int d\Gamma(\boldsymbol{k})\Big(\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}\frac{1}{\sqrt{N}}a_{\boldsymbol{a}}(\boldsymbol{k},1)\Big)\right)^{\otimes N}\exp\left(-\int d\Gamma(\boldsymbol{k})\Big(\alpha^{\boldsymbol{a}}(\boldsymbol{k})\frac{1}{\sqrt{N}}a_{\boldsymbol{a}}(\boldsymbol{k},1)^\dagger\Big)\right)^{\otimes N} \\
&\times \exp\left(\frac{1}{2}\int d\Gamma(\boldsymbol{k})\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}\alpha_{\boldsymbol{a}}(\boldsymbol{k})\frac{1}{N}I(\boldsymbol{k},1)\right)^{\otimes N} &(\text{G.5}) \\
&= \exp\left(\int d\Gamma(\boldsymbol{k})\Big(\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}\frac{1}{\sqrt{N}}a_{\boldsymbol{a}}(\boldsymbol{k},1) - \text{H.c.}\Big)\right)^{\otimes N} = \mathcal{D}\left(\frac{\alpha}{\sqrt{N}},1\right)^{\otimes N}. &(\text{G.6})
\end{aligned}$$

we obtain a coherent state

$$\mathcal{D}(\alpha,N)|O(N)\rangle = |\alpha(N)\rangle. \tag{G.7}$$

where

$$|\alpha(N)\rangle = \left(\exp\left(\int d\Gamma(\boldsymbol{k})\Big(\frac{1}{\sqrt{N}}\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}a_{\boldsymbol{a}}(\boldsymbol{k},1) - \text{H.c.}\Big)\right)|O(1)\rangle\right)^{\otimes N} = |\alpha(1)/\sqrt{N}\rangle^{\otimes N}. \tag{G.8}$$

Using Baker-Campbell-Hausdorff formula, we can rewrite coherent sates in the following useful forms

$$
\begin{aligned}
\mathcal{D}(\alpha,N) &= \exp\left(\int d\Gamma(\boldsymbol{k})\Big(\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}a_{\boldsymbol{a}}(\boldsymbol{k},N)-\mathrm{H.c.}\Big)\right) \\
&= \exp\left(\int d\Gamma(\boldsymbol{k})\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}a_{\boldsymbol{a}}(\boldsymbol{k},N)\right)\exp\left(-\int d\Gamma(\boldsymbol{k})\alpha^{\boldsymbol{a}}(\boldsymbol{k})a_{\boldsymbol{a}}(\boldsymbol{k},N)^\dagger\right) \\
&\times \exp\left(\frac{1}{2}\left[\int d\Gamma(\boldsymbol{k})\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}a_{\boldsymbol{a}}(\boldsymbol{k},N),-\int d\Gamma(\boldsymbol{k}')\alpha^{\boldsymbol{b}}(\boldsymbol{k}')a_{\boldsymbol{b}}(\boldsymbol{k}',N)^\dagger\right]\right) \\
&= \exp\left(\int d\Gamma(\boldsymbol{k})\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}a_{\boldsymbol{a}}(\boldsymbol{k},N)\right)\exp\left(-\int d\Gamma(\boldsymbol{k})\alpha^{\boldsymbol{a}}(\boldsymbol{k})a_{\boldsymbol{a}}(\boldsymbol{k},N)^\dagger\right)\exp\left(-\frac{1}{2}\int d\Gamma(\boldsymbol{k})\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}\alpha_{\boldsymbol{a}}(\boldsymbol{k})I(\boldsymbol{k},N)\right) \qquad \text{(G.9)} \\
&= \exp\left(-\int d\Gamma(\boldsymbol{k})\alpha^{\boldsymbol{a}}(\boldsymbol{k})a_{\boldsymbol{a}}(\boldsymbol{k},N)^\dagger\right)\exp\left(\int d\Gamma(\boldsymbol{k})\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}a_{\boldsymbol{a}}(\boldsymbol{k},N)\right)\exp\left(\frac{1}{2}\int d\Gamma(\boldsymbol{k})\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}\alpha_{\boldsymbol{a}}(\boldsymbol{k})I(\boldsymbol{k},N)\right) \qquad \text{(G.10)} \\
&= \exp\int d\Gamma(\boldsymbol{k})\Big(\alpha_1(\boldsymbol{k})a_1(\boldsymbol{k},N)^\dagger+\alpha_2(\boldsymbol{k})a_2(\boldsymbol{k},N)^\dagger+\alpha_3(\boldsymbol{k})a_3(\boldsymbol{k},N)^\dagger+\overline{\alpha_0(\boldsymbol{k})}a_0(\boldsymbol{k},N)\Big) \\
&\times \exp\int d\Gamma(\boldsymbol{k})\Big(-\overline{\alpha_1(\boldsymbol{k})}a_1(\boldsymbol{k},N)-\overline{\alpha_2(\boldsymbol{k})}a_2(\boldsymbol{k},N)-\overline{\alpha_3(\boldsymbol{k})}a_3(\boldsymbol{k},N)-\alpha_0(\boldsymbol{k})a_0(\boldsymbol{k},N)^\dagger\Big) \\
&\times \exp\left(-\frac{1}{2}\int d\Gamma(\boldsymbol{k})\Big(|\alpha_1(\boldsymbol{k})|^2+|\alpha_2(\boldsymbol{k})|^2+|\alpha_3(\boldsymbol{k})|^2+|\alpha_0(\boldsymbol{k})|^2\Big)I(\boldsymbol{k},N)\right), \qquad \text{(G.11)}
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{D}(\alpha,N)^\dagger a_{\boldsymbol{a}}(\boldsymbol{k},N)\mathcal{D}(\alpha,N) \\
&= \exp\left(\int d\Gamma(\boldsymbol{k})\Big(\alpha^{\boldsymbol{b}}(\boldsymbol{k})a_{\boldsymbol{b}}(\boldsymbol{k},N)^\dagger-\mathrm{H.c.}\Big)\right)a_{\boldsymbol{a}}(\boldsymbol{k},N)\exp\left(\int d\Gamma(\boldsymbol{k})\Big(\overline{\alpha^{\boldsymbol{b}}(\boldsymbol{k})}a_{\boldsymbol{b}}(\boldsymbol{k},N)-\mathrm{H.c.}\Big)\right) \\
&= a_{\boldsymbol{a}}(\boldsymbol{k},N)+\left[a_{\boldsymbol{a}}(\boldsymbol{k},N),\int d\Gamma(\boldsymbol{k}')\Big(\overline{\alpha^{\boldsymbol{b}}(\boldsymbol{k}')}a_{\boldsymbol{b}}(\boldsymbol{k}',N)-\mathrm{H.c.}\Big)\right] \\
&= a_{\boldsymbol{a}}(\boldsymbol{k},N)-\int d\Gamma(\boldsymbol{k}')\left[a_{\boldsymbol{a}}(\boldsymbol{k},N),\alpha^{\boldsymbol{b}}(\boldsymbol{k}')a_{\boldsymbol{b}}(\boldsymbol{k}',N)^\dagger\right] \\
&= a_{\boldsymbol{a}}(\boldsymbol{k},N)+g_{\boldsymbol{ab}}\ \alpha^{\boldsymbol{b}}(\boldsymbol{k})I(\boldsymbol{k},N)=a_{\boldsymbol{a}}(\boldsymbol{k},N)+\alpha_{\boldsymbol{a}}(\boldsymbol{k})I(\boldsymbol{k},N), \qquad \text{(G.12)}
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{D}(\alpha,N)^\dagger a_{\boldsymbol{a}}(\boldsymbol{k},N)^\dagger\mathcal{D}(\alpha,N) \\
&= \exp\left(\int d\Gamma(\boldsymbol{k})\Big(\alpha^{\boldsymbol{b}}(\boldsymbol{k})a_{\boldsymbol{b}}(\boldsymbol{k},N)^\dagger-\mathrm{H.c.}\Big)\right)a_{\boldsymbol{a}}(\boldsymbol{k},N)^\dagger\exp\left(\int d\Gamma(\boldsymbol{k})\Big(\overline{\alpha^{\boldsymbol{b}}(\boldsymbol{k})}a_{\boldsymbol{b}}(\boldsymbol{k},N)-\mathrm{H.c.}\Big)\right) \\
&= a_{\boldsymbol{a}}(\boldsymbol{k},N)^\dagger+\left[a_{\boldsymbol{a}}(\boldsymbol{k},N)^\dagger,\int d\Gamma(\boldsymbol{k}')\Big(\overline{\alpha^{\boldsymbol{b}}(\boldsymbol{k}')}a_{\boldsymbol{b}}(\boldsymbol{k}',N)-\mathrm{H.c.}\Big)\right] \\
&= a_{\boldsymbol{a}}(\boldsymbol{k},N)^\dagger+\int d\Gamma(\boldsymbol{k}')\left[a_{\boldsymbol{a}}(\boldsymbol{k},N)^\dagger,\overline{\alpha^{\boldsymbol{b}}(\boldsymbol{k}')}a_{\boldsymbol{b}}(\boldsymbol{k}',N)\right] \\
&= a_{\boldsymbol{a}}(\boldsymbol{k},N)^\dagger+g_{\boldsymbol{ab}}\ \overline{\alpha^{\boldsymbol{b}}(\boldsymbol{k})}I(\boldsymbol{k},N)=a_{\boldsymbol{a}}(\boldsymbol{k},N)^\dagger+\overline{\alpha_{\boldsymbol{a}}(\boldsymbol{k})}I(\boldsymbol{k},N), \qquad \text{(G.13)}
\end{aligned}
$$

$$
\begin{aligned}
&a_{\boldsymbol{a}}(\boldsymbol{k},N)|\alpha,N\rangle \\
&= \mathcal{D}(\alpha,N)\mathcal{D}(\alpha,N)^\dagger a_{\boldsymbol{a}}(\boldsymbol{k},N)\mathcal{D}(\alpha,N)|O(N)\rangle \\
&= \mathcal{D}(\alpha,N)\left(a_{\boldsymbol{a}}(\boldsymbol{k},N)+\alpha_{\boldsymbol{a}}(\boldsymbol{k})I(\boldsymbol{k},N)\right)|O(N)\rangle \\
&= \alpha_{\boldsymbol{a}}(\boldsymbol{k})I(\boldsymbol{k},N)|\alpha,N\rangle, \qquad \text{(G.14)}
\end{aligned}
$$

$$
\begin{aligned}
& [a_{\boldsymbol{a}}(\boldsymbol{k},N),\mathcal{D}(\alpha,N)] \\
= \quad & \left[a_{\boldsymbol{a}}(\boldsymbol{k},N),\exp\left(\int d\Gamma(\boldsymbol{k})\Big(\overline{\alpha^{\boldsymbol{b}}(\boldsymbol{k})}a_{\boldsymbol{b}}(\boldsymbol{k},N)-\text{H.c.}\Big)\right)\right] \\
= \quad & \exp\left(\int d\Gamma(\boldsymbol{k})\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}a_{\boldsymbol{a}}(\boldsymbol{k},N)\right)\exp\left(-\frac{1}{2}\int d\Gamma(\boldsymbol{k})\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}\alpha_{\boldsymbol{a}}(\boldsymbol{k})I(\boldsymbol{k},N)\right) \\
\times \quad & \left[a_{\boldsymbol{a}}(\boldsymbol{k},N),\exp\left(-\int d\Gamma(\boldsymbol{k})\alpha^{\boldsymbol{a}}(\boldsymbol{k})a_{\boldsymbol{a}}(\boldsymbol{k},N)^\dagger\right)\right] \\
= \quad & \exp\left(\int d\Gamma(\boldsymbol{k})\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}a_{\boldsymbol{a}}(\boldsymbol{k},N)\right)\exp\left(-\frac{1}{2}\int d\Gamma(\boldsymbol{k})\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}\alpha_{\boldsymbol{a}}(\boldsymbol{k})I(\boldsymbol{k},N)\right) \\
\times \quad & \sum_{n=0}^{\infty}\frac{1}{n!}\left[a_{\boldsymbol{a}}(\boldsymbol{k},N),\left(-\int d\Gamma(\boldsymbol{k})\alpha^{\boldsymbol{b}}(\boldsymbol{k})a_{\boldsymbol{b}}(\boldsymbol{k},N)^\dagger\right)^n\right] \\
= \quad & \exp\left(\int d\Gamma(\boldsymbol{k})\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}a_{\boldsymbol{a}}(\boldsymbol{k},N)\right)\exp\left(-\frac{1}{2}\int d\Gamma(\boldsymbol{k})\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}\alpha_{\boldsymbol{a}}(\boldsymbol{k})I(\boldsymbol{k},N)\right) \\
\times \quad & \alpha_{\boldsymbol{a}}(\boldsymbol{k})\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\left(-\int d\Gamma(\boldsymbol{k})\alpha^{\boldsymbol{b}}(\boldsymbol{k})a_{\boldsymbol{b}}(\boldsymbol{k},N)^\dagger\right)^{n-1} \\
= \quad & \alpha_{\boldsymbol{a}}(\boldsymbol{k})\mathcal{D}(\alpha,N),
\end{aligned}
\tag{G.15}
$$

$$
\begin{aligned}
& [a_{\boldsymbol{a}}(\boldsymbol{k},N)^\dagger,\mathcal{D}(\alpha,N)] \;=\; -\,[a_{\boldsymbol{a}}(\boldsymbol{k},N),\mathcal{D}(-\alpha,N)]^\dagger \;=\; -\,(\alpha_{\boldsymbol{a}}(\boldsymbol{k},N)\mathcal{D}(-\alpha,N))^\dagger \\
= \quad & -\overline{\alpha_{\boldsymbol{a}}(\boldsymbol{k})}\mathcal{D}(\alpha,N),
\end{aligned}
\tag{G.16}
$$

$$
\begin{aligned}
& \mathcal{D}(\alpha,N)\mathcal{D}(\beta,N) \\
= \quad & \exp\left(\int d\Gamma(\boldsymbol{k})\Big(\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}a_{\boldsymbol{a}}(\boldsymbol{k},N)-\text{H.c.}\Big)\right)\exp\left(\int d\Gamma(\boldsymbol{k})\Big(\overline{\beta^{\boldsymbol{b}}(\boldsymbol{k})}a_{\boldsymbol{b}}(\boldsymbol{k},N)-\text{H.c.}\Big)\right) \\
= \quad & \exp\left(\int d\Gamma(\boldsymbol{k})\Big(\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}a_{\boldsymbol{a}}(\boldsymbol{k},N)+\overline{\beta^{\boldsymbol{b}}(\boldsymbol{k})}a_{\boldsymbol{b}}(\boldsymbol{k},N)-\text{H.c.}\Big)\right) \\
\times \quad & \exp\frac{1}{2}\left[\int d\Gamma(\boldsymbol{k})\Big(\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}a_{\boldsymbol{a}}(\boldsymbol{k},N)-\text{H.c.}\Big),\int d\Gamma(\boldsymbol{k})\Big(\overline{\beta^{\boldsymbol{b}}(\boldsymbol{k})}a_{\boldsymbol{b}}(\boldsymbol{k},N)-\text{H.c.}\Big)\right] \\
= \quad & \exp\left(\int d\Gamma(\boldsymbol{k})\Big((\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}+\overline{\beta^{\boldsymbol{a}}(\boldsymbol{k})})a_{\boldsymbol{a}}(\boldsymbol{k},N)-\text{H.c.}\Big)\right) \\
\times \quad & \exp\frac{1}{2}\left(\int d\Gamma(\boldsymbol{k})\Big(\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}\beta_{\boldsymbol{a}}(\boldsymbol{k})-\text{H.c.}\Big)I(\boldsymbol{k},N)\right) \\
= \quad & \mathcal{D}(\alpha+\beta,N)\times\exp\frac{1}{2}\left(\int d\Gamma(\boldsymbol{k})\Big(\overline{\alpha^{\boldsymbol{a}}(\boldsymbol{k})}\beta_{\boldsymbol{a}}(\boldsymbol{k})-\text{H.c.}\Big)I(\boldsymbol{k},N)\right).
\end{aligned}
\tag{G.17}
$$

H Bell states transformation

Most of the calculations done in this appendix will be used for chapter 7. First let us remind ourselves that the correspondence between annihilation and creation operators in linear and circular basis may be written as

$$
\begin{aligned}
a_s(\boldsymbol{k},N) &= \frac{1}{\sqrt{2}}e^{-is\theta(\boldsymbol{k})}\left(a_\theta(\boldsymbol{k},N)-isa_{\theta'}(\boldsymbol{k},N)\right),\\
a_\theta(\boldsymbol{k},N) &= \frac{1}{\sqrt{2}}\sum_{s=\pm}a_s(\boldsymbol{k},N)e^{is\theta(\boldsymbol{k})}.
\end{aligned}
$$

For the $N=1$ oscillator representation we have the following correspondence between creation operators in both polarization basis:

$$
\begin{aligned}
a_s(\boldsymbol{k},1)^\dagger a_{s'}(\boldsymbol{k}',1)^\dagger &= \frac{1}{2}e^{i(s\theta(\boldsymbol{k})+s'\theta(\boldsymbol{k}'))}\left(a_\theta(\boldsymbol{k},1)^\dagger a_\theta(\boldsymbol{k}',1)^\dagger - ss'a_{\theta'}(\boldsymbol{k},1)^\dagger a_{\theta'}(\boldsymbol{k}',1)^\dagger\right)\\
&+ \frac{1}{2}e^{i(s\theta(\boldsymbol{k})+s'\theta(\boldsymbol{k}'))}\left(is'a_\theta(\boldsymbol{k},1)^\dagger a_{\theta'}(\boldsymbol{k}',1)^\dagger + isa_{\theta'}(\boldsymbol{k},1)^\dagger a_\theta(\boldsymbol{k}',1)^\dagger\right).
\end{aligned}
\tag{H.1}
$$

To describe such correspondence between the two basis in two-photon fields for N-oscillator representation we will need the following formulas

$$
\begin{aligned}
& a_s(\boldsymbol{k},N)^\dagger a_{s'}(\boldsymbol{k}',N)^\dagger\\
=\ & \frac{1}{N}\sum_{nm}^{N}a_s(\boldsymbol{k},1)^{\dagger(n)}a_{s'}(\boldsymbol{k}',1)^{\dagger(m)}\\
=\ & \frac{1}{N}\sum_{nm}^{N}\left(\frac{1}{\sqrt{2}}e^{is\theta(\boldsymbol{k})}\left(a_\theta(\boldsymbol{k},1)^\dagger + isa_{\theta'}(\boldsymbol{k},1)^\dagger\right)\right)^{(n)}\left(\frac{1}{\sqrt{2}}e^{is'\theta(\boldsymbol{k}')}\left(a_\theta(\boldsymbol{k}',1)^\dagger + is'a_{\theta'}(\boldsymbol{k}',1)^\dagger\right)\right)^{(m)}\\
=\ & \frac{1}{N}\frac{1}{2}e^{i(s\theta(\boldsymbol{k})+s'\theta(\boldsymbol{k}'))}\sum_{nm}^{N}\left(a_\theta(\boldsymbol{k},1)^{\dagger(n)} + isa_{\theta'}(\boldsymbol{k},1)^{\dagger(n)}\right)\left(a_\theta(\boldsymbol{k}',1)^{\dagger(m)} + is'a_{\theta'}(\boldsymbol{k}',1)^{\dagger(m)}\right)\\
=\ & \frac{1}{2}e^{i(s\theta(\boldsymbol{k})+s'\theta(\boldsymbol{k}'))}\left(a_\theta(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger - ss'a_{\theta'}(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger\right)\\
+\ & \frac{1}{2}e^{i(s\theta(\boldsymbol{k})+s'\theta(\boldsymbol{k}'))}\left(is'a_\theta(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger + isa_{\theta'}(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger\right),
\end{aligned}
\tag{H.2}
$$

This formula can be written explicitly for left and right handed polarization creation operators:

$$
\begin{aligned}
& a_+(\boldsymbol{k},N)^\dagger a_+(\boldsymbol{k}',N)^\dagger\\
=\ & \frac{1}{2}e^{i(\theta(\boldsymbol{k})+\theta(\boldsymbol{k}'))}\left(a_\theta(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger - a_{\theta'}(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger\right)\\
+\ & \frac{1}{2}e^{i(\theta(\boldsymbol{k})+\theta(\boldsymbol{k}'))}\left(ia_\theta(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger + ia_{\theta'}(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger\right),
\end{aligned}
\tag{H.3}
$$

$$
\begin{aligned}
& a_-(\boldsymbol{k},N)^\dagger a_-(\boldsymbol{k}',N)^\dagger\\
=\ & \frac{1}{2}e^{-i(\theta(\boldsymbol{k})+\theta(\boldsymbol{k}'))}\left(a_\theta(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger - a_{\theta'}(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger\right)\\
-\ & \frac{1}{2}e^{-i(\theta(\boldsymbol{k})+\theta(\boldsymbol{k}'))}\left(ia_\theta(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger + ia_{\theta'}(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger\right),
\end{aligned}
\tag{H.4}
$$

$$
\begin{aligned}
& a_+(\boldsymbol{k},N)^\dagger a_-(\boldsymbol{k}',N)^\dagger\\
=\ & \frac{1}{2}e^{i(\theta(\boldsymbol{k})-\theta(\boldsymbol{k}'))}\left(a_\theta(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger + a_{\theta'}(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger\right)\\
-\ & \frac{1}{2}e^{i(\theta(\boldsymbol{k})-\theta(\boldsymbol{k}'))}\left(ia_\theta(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger - ia_{\theta'}(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger\right),
\end{aligned}
\tag{H.5}
$$

$$
\begin{aligned}
& a_-(\boldsymbol{k},N)^\dagger a_+(\boldsymbol{k}',N)^\dagger\\
=\ & \frac{1}{2}e^{-i(\theta(\boldsymbol{k})-\theta(\boldsymbol{k}'))}\left(a_\theta(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger + a_{\theta'}(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger\right)\\
+\ & \frac{1}{2}e^{-i(\theta(\boldsymbol{k})-\theta(\boldsymbol{k}'))}\left(ia_\theta(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k}',N)^\dagger - ia_{\theta'}(\boldsymbol{k},N)^\dagger a_\theta(\boldsymbol{k}',N)^\dagger\right).
\end{aligned}
\tag{H.6}
$$

We will also need an inverse correspondence, i.e.

$$a_\theta(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k'},N)^\dagger = \frac{1}{2}\sum_{ss'=\pm} e^{-is'\pi/2} a_s(\boldsymbol{k},N)^\dagger a_{s'}(\boldsymbol{k'},N)^\dagger e^{-is\theta(\boldsymbol{k})} e^{-is'\theta(\boldsymbol{k'})}, \tag{H.7}$$

$$a_\theta(\boldsymbol{k},N)^\dagger a_{\theta}(\boldsymbol{k'},N)^\dagger = \frac{1}{2}\sum_{ss'=\pm} a_s(\boldsymbol{k},N)^\dagger a_{s'}(\boldsymbol{k'},N)^\dagger e^{-is\theta(\boldsymbol{k})} e^{-is'\theta(\boldsymbol{k'})}, \tag{H.8}$$

$$a_{\theta'}(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k'},N)^\dagger = \frac{1}{2}\sum_{ss'=\pm} e^{-i(s+s')\pi/2} a_s(\boldsymbol{k},N)^\dagger a_{s'}(\boldsymbol{k'},N)^\dagger e^{-is\theta(\boldsymbol{k})} e^{-is'\theta(\boldsymbol{k'})}. \tag{H.9}$$

The following calculations show that all Bell states transform under Lorentz transformation (385) as scalars independent on the basis they are considered in. In the calculus one has to take into account the transformation rule for the polarization angle and its shift due to Wigner phase, so that

$$\begin{aligned}
& U(\Lambda,0,N) a_\theta(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k'},N)^\dagger U(\Lambda,0,N)^\dagger \\
= & \frac{1}{2} U(\Lambda,0,N) \sum_{ss'=\pm} e^{-is'\pi/2} a_s(\boldsymbol{k},N)^\dagger a_{s'}(\boldsymbol{k'},N)^\dagger e^{-is\theta(\boldsymbol{k})} e^{-is'\theta(\boldsymbol{k'})} U(\Lambda,0,N)^\dagger \\
= & \frac{1}{2}\sum_{ss'=\pm} e^{-is'\pi/2} e^{-2is\Theta(\Lambda,\boldsymbol{\Lambda k})} e^{-2is'\Theta(\Lambda,\boldsymbol{\Lambda k'})} a_s(\boldsymbol{\Lambda k},N)^\dagger a_{s'}(\boldsymbol{\Lambda k'},N)^\dagger e^{-is\theta(\boldsymbol{k})} e^{-is'\theta(\boldsymbol{k'})} \\
= & \frac{1}{2}\sum_{ss'=\pm} e^{-is'\pi/2} e^{-2is\Theta(\Lambda,\boldsymbol{\Lambda k})} e^{-2is'\Theta(\Lambda,\boldsymbol{\Lambda k'})} a_s(\boldsymbol{\Lambda k},N)^\dagger a_{s'}(\boldsymbol{\Lambda k'},N)^\dagger e^{-is(\theta(\boldsymbol{\Lambda k})-2\Theta(\Lambda,\boldsymbol{\Lambda k}))} e^{-is'(\theta(\boldsymbol{\Lambda k'})-2\Theta(\Lambda,\boldsymbol{\Lambda k'}))} \\
= & \frac{1}{2}\sum_{ss'=\pm} e^{-is'\pi/2} a_s(\boldsymbol{\Lambda k},N)^\dagger a_{s'}(\boldsymbol{\Lambda k'},N)^\dagger e^{-is\theta(\boldsymbol{\Lambda k})} e^{-is'\theta(\boldsymbol{\Lambda k'})} \\
= & a_\theta(\boldsymbol{\Lambda k},N)^\dagger a_{\theta'}(\boldsymbol{\Lambda k'},N)^\dagger,
\end{aligned} \tag{H.10}$$

$$\begin{aligned}
& U(\Lambda,0,N) a_\theta(\boldsymbol{k},N)^\dagger a_{\theta}(\boldsymbol{k'},N)^\dagger U(\Lambda,0,N)^\dagger \\
= & \frac{1}{2} U(\Lambda,0,N) \sum_{ss'=\pm} a_s(\boldsymbol{k},N)^\dagger a_{s'}(\boldsymbol{k'},N)^\dagger e^{-is\theta(\boldsymbol{k})} e^{-is'\theta(\boldsymbol{k'})} U(\Lambda,0,N)^\dagger \\
= & \frac{1}{2}\sum_{ss'=\pm} e^{-2is\Theta(\Lambda,\boldsymbol{\Lambda k})} e^{-2is'\Theta(\Lambda,\boldsymbol{\Lambda k'})} a_s(\boldsymbol{\Lambda k},N)^\dagger a_{s'}(\boldsymbol{\Lambda k'},N)^\dagger e^{-is\theta(\boldsymbol{k})} e^{-is'\theta(\boldsymbol{k'})} \\
= & \frac{1}{2}\sum_{ss'=\pm} e^{-2is\Theta(\Lambda,\boldsymbol{\Lambda k})} e^{-2is'\Theta(\Lambda,\boldsymbol{\Lambda k'})} a_s(\boldsymbol{\Lambda k},N)^\dagger a_{s'}(\boldsymbol{\Lambda k'},N)^\dagger e^{-is(\theta(\boldsymbol{\Lambda k})-2\Theta(\Lambda,\boldsymbol{\Lambda k}))} e^{-is'(\theta(\boldsymbol{\Lambda k'})-2\Theta(\Lambda,\boldsymbol{\Lambda k'}))} \\
= & \frac{1}{2}\sum_{ss'=\pm} a_s(\boldsymbol{\Lambda k},N)^\dagger a_{s'}(\boldsymbol{\Lambda k'},N)^\dagger e^{-is(\theta(\boldsymbol{\Lambda k}))} e^{-is'(\theta(\boldsymbol{\Lambda k'}))} \\
= & a_\theta(\boldsymbol{\Lambda k},N)^\dagger a_{\theta}(\boldsymbol{\Lambda k'},N)^\dagger,
\end{aligned} \tag{H.11}$$

$$\begin{aligned}
& U(\Lambda,0,N) a_{\theta'}(\boldsymbol{k},N)^\dagger a_{\theta'}(\boldsymbol{k'},N)^\dagger U(\Lambda,0,N)^\dagger \\
= & \frac{1}{2} U(\Lambda,0,N) \sum_{ss'=\pm} e^{-i(s+s')\pi/2} a_s(\boldsymbol{k},N)^\dagger a_{s'}(\boldsymbol{k'},N)^\dagger e^{-is\theta(\boldsymbol{k})} e^{-is'\theta(\boldsymbol{k'})} U(\Lambda,0,N)^\dagger \\
= & \frac{1}{2}\sum_{ss'=\pm} e^{-i(s+s')\pi/2} e^{-2is\Theta(\Lambda,\boldsymbol{\Lambda k})} e^{-2is'\Theta(\Lambda,\boldsymbol{\Lambda k'})} a_s(\boldsymbol{\Lambda k},N)^\dagger a_{s'}(\boldsymbol{\Lambda k'},N)^\dagger e^{-is\theta(\boldsymbol{k})} e^{-is'\theta(\boldsymbol{k'})} \\
= & \frac{1}{2}\sum_{ss'=\pm} e^{-i(s+s')\pi/2} e^{-2is\Theta(\Lambda,\boldsymbol{\Lambda k})} e^{-2is'\Theta(\Lambda,\boldsymbol{\Lambda k'})} a_s(\boldsymbol{\Lambda k},N)^\dagger a_{s'}(\boldsymbol{\Lambda k'},N)^\dagger e^{-is(\theta(\boldsymbol{\Lambda k})-2\Theta(\Lambda,\boldsymbol{\Lambda k}))} e^{-is'(\theta(\boldsymbol{\Lambda k'})-2\Theta(\Lambda,\boldsymbol{\Lambda k'}))} \\
= & \frac{1}{2}\sum_{ss'=\pm} e^{-i(s+s')\pi/2} a_s(\boldsymbol{\Lambda k},N)^\dagger a_{s'}(\boldsymbol{\Lambda k'},N)^\dagger e^{-is(\theta(\boldsymbol{\Lambda k}))} e^{-is'(\theta(\boldsymbol{\Lambda k'}))} \\
= & a_{\theta'}(\boldsymbol{\Lambda k},N)^\dagger a_{\theta'}(\boldsymbol{\Lambda k'},N)^\dagger.
\end{aligned} \tag{H.12}$$

The following formula is calculated step by step for (568) in section 9.1

$$
\begin{aligned}
& U(\Lambda,0,N)^\dagger Y_\alpha(\boldsymbol{l},N)U(\Lambda,0,N) \\
= \;& U(\Lambda,0,N)^\dagger \sum_{n=1}^{N}\sum_{s=\pm}\left(e^{2is\alpha}|\boldsymbol{l}\rangle\langle\boldsymbol{l}|\otimes a_{-s}^\dagger a_s\right)^{(n)} U(\Lambda,0,N) \\
= \;& \sum_{n=1}^{N}\sum_{s=\pm}\left(U(\Lambda,0,1)^\dagger e^{2is\alpha}|\boldsymbol{l}\rangle\langle\boldsymbol{l}|\otimes a_{-s}^\dagger a_s U(\Lambda,0,1)\right)^{(n)} \\
= \;& \sum_{n=1}^{N}\sum_{s=\pm}\left(\left(\int d\Gamma(\boldsymbol{k})\int d\Gamma(\boldsymbol{k}')|\boldsymbol{\Lambda^{-1}k}\rangle\langle\boldsymbol{k}|\boldsymbol{l}\rangle\langle\boldsymbol{l}|\boldsymbol{k}'\rangle\langle\boldsymbol{\Lambda^{-1}k'}|\right)\otimes\left(U(\Lambda,\boldsymbol{k})^\dagger e^{2is\alpha}a_{-s}^\dagger a_s U(\Lambda,\boldsymbol{k}')\right)\right)^{(n)} \\
= \;& \sum_{n=1}^{N}\sum_{s=\pm}\left(|\boldsymbol{\Lambda^{-1}l}\rangle\langle\boldsymbol{\Lambda^{-1}l}|\otimes\left(U(\Lambda,\boldsymbol{l})^\dagger e^{2is\alpha}a_{-s}^\dagger a_s U(\Lambda,\boldsymbol{l})\right)\right)^{(n)} \\
= \;& \sum_{n=1}^{N}\sum_{s=\pm}\left(e^{2is\alpha}e^{-4is\Theta(\Lambda,\boldsymbol{l})}|\boldsymbol{\Lambda^{-1}l}\rangle\langle\boldsymbol{\Lambda^{-1}l}|\otimes a_{-s}^\dagger a_s\right)^{(n)} \;=\; Y_{\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})}(\boldsymbol{\Lambda^{-1}l},N). \qquad \text{(H.13)}
\end{aligned}
$$

I Explicit calculations of commutation relations

This formula is derived for (80) in section 2.4

$$\begin{aligned}
[a_\alpha(\boldsymbol{k},N), a_{\alpha'}(\boldsymbol{k'},N)^\dagger] &= \frac{1}{N}\sum_{m,n=1}^{N}[a_\alpha(\boldsymbol{k},1)^{(m)}, a_{\alpha'}(\boldsymbol{k'},1)^{\dagger(n)}] \\
&= \frac{1}{N}\sum_{m,n=1}^{N}[a_\alpha(\boldsymbol{k},1), a_{\alpha'}(\boldsymbol{k'},1)^\dagger]^{(n)}\delta_{m,n} \\
&= \frac{1}{N}\sum_{m,n=1}^{N}\left(\delta_{\alpha,\alpha'}\delta_\Gamma(\boldsymbol{k},\boldsymbol{k'})I(\boldsymbol{k},1)\right)^{(n)}\delta_{m,n} \\
&= \delta_{\alpha,\alpha'}\delta_\Gamma(\boldsymbol{k},\boldsymbol{k'})\frac{1}{N}\sum_{n}^{N}I(\boldsymbol{k},1)^{(n)} \\
&= \delta_{\alpha,\alpha'}\delta_\Gamma(\boldsymbol{k},\boldsymbol{k'})I(\boldsymbol{k},N),
\end{aligned} \tag{I.1}$$

$$\begin{aligned}
&\left[a_s(\boldsymbol{k},N)a_{s'}(\boldsymbol{k'},N), a_r(\boldsymbol{l},N)^\dagger a_{r'}(\boldsymbol{l'},N)^\dagger\right] \\
=\ & a_s(\boldsymbol{k},N)a_{r'}(\boldsymbol{l'},N)^\dagger I(\boldsymbol{l},N)\delta_{r,s'}\delta_\Gamma(\boldsymbol{l},\boldsymbol{k'}) + a_{s'}(\boldsymbol{k'},N)a_{r'}(\boldsymbol{l'},N)^\dagger I(\boldsymbol{l},N)\delta_{s,r}\delta_\Gamma(\boldsymbol{k},\boldsymbol{l}) \\
+\ & a_r(\boldsymbol{l},N)^\dagger a_s(\boldsymbol{k},N)I(\boldsymbol{l'},N)\delta_{s',r'}\delta_\Gamma(\boldsymbol{k'},\boldsymbol{l'}) + a_r(\boldsymbol{l},N)^\dagger a_{s'}(\boldsymbol{k'},N)I(\boldsymbol{l'},N)\delta_{r',s}\delta_\Gamma(\boldsymbol{l'},\boldsymbol{k}).
\end{aligned} \tag{I.2}$$

The following commutation relations are calculated using (I.2)

$$\begin{aligned}
& [\Psi(N), \Psi(N)^\dagger] \\
=\ & \sum_{rr'=\pm}\sum_{s,s'=\pm}\left[\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k'})\ \psi_{ss'}(\boldsymbol{k},\boldsymbol{k'})a_s(\boldsymbol{k},N)^\dagger a_{s'}(\boldsymbol{k'},N)^\dagger, \int d\Gamma(\boldsymbol{l})d\Gamma(\boldsymbol{l'})\ \bar{\psi}_{rr'}(\boldsymbol{l},\boldsymbol{l'})a_r(\boldsymbol{l},N)a_{r'}(\boldsymbol{l'},N)\right] \\
=\ & \sum_{rr'=\pm}\sum_{s,s'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k'})d\Gamma(\boldsymbol{l})d\Gamma(\boldsymbol{l'})\ \psi_{ss'}(\boldsymbol{k},\boldsymbol{k'})\bar{\psi}_{rr'}(\boldsymbol{l},\boldsymbol{l'})\ [a_s(\boldsymbol{k},N)^\dagger a_{s'}(\boldsymbol{k'},N)^\dagger, a_r(\boldsymbol{l},N)a_{r'}(\boldsymbol{l'},N)] \\
=\ & -\sum_{rr'=\pm}\sum_{s,s'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k'})d\Gamma(\boldsymbol{l})d\Gamma(\boldsymbol{l'})\ \psi_{ss'}(\boldsymbol{k},\boldsymbol{k'})\bar{\psi}_{rr'}(\boldsymbol{l},\boldsymbol{l'}) \\
\times\ & \big(a_r(\boldsymbol{l},N)a_{s'}(\boldsymbol{k'},N)^\dagger I(\boldsymbol{k},N)\delta_{s,r'}\delta_\Gamma(\boldsymbol{k},\boldsymbol{l'}) + a_{r'}(\boldsymbol{l'},N)a_{s'}(\boldsymbol{k'},N)^\dagger I(\boldsymbol{k},N)\delta_{s,r}\delta_\Gamma(\boldsymbol{k},\boldsymbol{l}) \\
+\ & a_s(\boldsymbol{k},N)^\dagger a_r(\boldsymbol{l},N)I(\boldsymbol{k'},N)\delta_{s',r'}\delta_\Gamma(\boldsymbol{k'},\boldsymbol{l'}) + a_s(\boldsymbol{k},N)^\dagger a_{r'}(\boldsymbol{l'},N)I(\boldsymbol{k'},N)\delta_{s',r}\delta_\Gamma(\boldsymbol{k'},\boldsymbol{l})\big)\,,
\end{aligned} \tag{I.3}$$

$$\begin{aligned}
& [\Psi_1(N), \Psi_1(N)^\dagger] \\
=\ & \sum_{r=\pm}\sum_{s=\pm}\left[\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k'})\ \psi_{s-s}(\boldsymbol{k},\boldsymbol{k'})a_s(\boldsymbol{k},N)^\dagger a_{-s}(\boldsymbol{k'},N)^\dagger, \int d\Gamma(\boldsymbol{l})d\Gamma(\boldsymbol{l'})\ \bar{\psi}_{r-r}(\boldsymbol{l},\boldsymbol{l'})a_r(\boldsymbol{l},N)a_{-r}(\boldsymbol{l'},N)\right] \\
=\ & -\sum_{r=\pm}\sum_{s=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k'})d\Gamma(\boldsymbol{l})d\Gamma(\boldsymbol{l'})\ \psi_{s-s}(\boldsymbol{k},\boldsymbol{k'})\bar{\psi}_{r-r}(\boldsymbol{l},\boldsymbol{l'}) \\
\times\ & \big(a_r(\boldsymbol{l},N)a_{-s}(\boldsymbol{k'},N)^\dagger I(\boldsymbol{k},N)\delta_{s,-r}\delta_\Gamma(\boldsymbol{k},\boldsymbol{l'}) + a_{-r}(\boldsymbol{l'},N)a_{-s}(\boldsymbol{k'},N)^\dagger I(\boldsymbol{k},N)\delta_{s,r}\delta_\Gamma(\boldsymbol{k},\boldsymbol{l}) \\
+\ & a_s(\boldsymbol{k},N)^\dagger a_r(\boldsymbol{l},N)I(\boldsymbol{k'},N)\delta_{s,r}\delta_\Gamma(\boldsymbol{k'},\boldsymbol{l'}) + a_s(\boldsymbol{k},N)^\dagger a_{-r}(\boldsymbol{l'},N)I(\boldsymbol{k'},N)\delta_{-s,r}\delta_\Gamma(\boldsymbol{k'},\boldsymbol{l})\big)\,,
\end{aligned} \tag{I.4}$$

$$\begin{aligned}
& [\Psi_2(N), \Psi_2(N)^\dagger] \\
=\ & \sum_{r=\pm}\sum_{s=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k'})d\Gamma(\boldsymbol{l})d\Gamma(\boldsymbol{l'})\ \psi_{ss}(\boldsymbol{k},\boldsymbol{k'})\bar{\psi}_{rr}(\boldsymbol{l},\boldsymbol{l'})\ \left[a_s(\boldsymbol{k},N)^\dagger a_s(\boldsymbol{k'},N)^\dagger, a_r(\boldsymbol{l},N)a_r(\boldsymbol{l'},N)\right] \\
=\ & -\sum_{r=\pm}\sum_{s=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k'})d\Gamma(\boldsymbol{l})d\Gamma(\boldsymbol{l'})\ \psi_{ss}(\boldsymbol{k},\boldsymbol{k'})\bar{\psi}_{rr}(\boldsymbol{l},\boldsymbol{l'}) \\
\times\ & \big(a_r(\boldsymbol{l},N)a_s(\boldsymbol{k'},N)^\dagger I(\boldsymbol{k},N)\delta_{s,r}\delta_\Gamma(\boldsymbol{k},\boldsymbol{l'}) + a_r(\boldsymbol{l'},N)a_s(\boldsymbol{k'},N)^\dagger I(\boldsymbol{k},N)\delta_{s,r}\delta_\Gamma(\boldsymbol{k},\boldsymbol{l}) \\
+\ & a_s(\boldsymbol{k},N)^\dagger a_r(\boldsymbol{l},N)I(\boldsymbol{k'},N)\delta_{s,r}\delta_\Gamma(\boldsymbol{k'},\boldsymbol{l'}) + a_s(\boldsymbol{k},N)^\dagger a_r(\boldsymbol{l'},N)I(\boldsymbol{k'},N)\delta_{s,r}\delta_\Gamma(\boldsymbol{k'},\boldsymbol{l})\big)\,.
\end{aligned} \tag{I.5}$$

The following two formulas are calculated explicitly for (541) and (542) in section 8.2

$$
\begin{aligned}
& [Y_\alpha(\boldsymbol{l},N),\Psi(N)] \\
= & \sum_{n=1}^{N}\sum_{r,s,s'=\pm}\left[\left(e^{2ir\alpha}|\boldsymbol{l}\rangle\langle\boldsymbol{l}|\otimes a_{-r}^\dagger a_r\right)^{(n)},\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k'})\ \psi_{ss'}(\boldsymbol{k},\boldsymbol{k'})a_s(\boldsymbol{k},N)^\dagger a_{s'}(\boldsymbol{k'},N)^\dagger\right] \\
= & \frac{1}{N}\sum_{n,m,p=1}^{N}\sum_{r,s,s'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k'})\ e^{2ir\alpha}\psi_{ss'}(\boldsymbol{k},\boldsymbol{k'})\left[(|\boldsymbol{l}\rangle\langle\boldsymbol{l}|\otimes a_{-r}^\dagger a_r)^{(n)},a_s(\boldsymbol{k},1)^{\dagger(m)}a_{s'}(\boldsymbol{k'},1)^{\dagger(p)}\right] \\
= & \frac{1}{N}\sum_{n,m,p=1}^{N}\sum_{r,s,s'=\pm}e^{2ir\alpha}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k'})\ \psi_{ss'}(\boldsymbol{k},\boldsymbol{k'}) \\
\times & \Big((|\boldsymbol{l}\rangle\langle\boldsymbol{k}|\otimes a_{-r}^\dagger)^{(n)}a_{s'}(\boldsymbol{k'},1)^{\dagger(p)}\delta_\Gamma(\boldsymbol{l},\boldsymbol{k})\delta_{r,s}\delta_{n,m}(|\boldsymbol{l}\rangle\langle\boldsymbol{k'}|\otimes a_{-r}^\dagger)^{(n)}a_s(\boldsymbol{k},1)^{\dagger(m)}\delta_\Gamma(\boldsymbol{l},\boldsymbol{k'})\delta_{r,s'}\delta_{n,p}\Big) \\
= & \ 2\sum_{s,s'=\pm}e^{2is\alpha}\int d\Gamma(\boldsymbol{k})\ \psi_{ss'}(\boldsymbol{l},\boldsymbol{k})a_{-s}(\boldsymbol{l},N)^\dagger a_{s'}(\boldsymbol{k},N)^\dagger,
\end{aligned}
\tag{I.6}
$$

$$
\begin{aligned}
& [Y_\alpha(\boldsymbol{l},N),\Psi(N)^\dagger] \\
= & \sum_{n=1}^{N}\sum_{r=\pm}\sum_{s,s'=\pm}\left[\left(e^{2ir\alpha}|\boldsymbol{l}\rangle\langle\boldsymbol{l}|\otimes a_{-r}^\dagger a_r\right)^{(n)},\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k'})\ \bar{\psi}_{ss'}(\boldsymbol{k},\boldsymbol{k'})a_s(\boldsymbol{k},N)a_{s'}(\boldsymbol{k'},N)\right] \\
= & \frac{1}{N}\sum_{n,m,p=1}^{N}\sum_{r=\pm}\sum_{s,s'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k'})\ e^{2ir\alpha}\bar{\psi}_{ss'}(\boldsymbol{k},\boldsymbol{k'})\left[(|\boldsymbol{l}\rangle\langle\boldsymbol{l}|\otimes a_{-r}^\dagger a_r)^{(n)},a_s(\boldsymbol{k},1)^{(m)}a_{s'}(\boldsymbol{k'},1)^{(p)}\right] \\
= & -\frac{1}{N}\sum_{n,m,p=1}^{N}\sum_{r=\pm}\sum_{s,s'=\pm}e^{2ir\alpha}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k'})\ \bar{\psi}_{ss'}(\boldsymbol{k},\boldsymbol{k'}) \\
\times & \Big((|\boldsymbol{l}\rangle\langle\boldsymbol{k'}|\otimes a_r(\boldsymbol{l},1))^{(n)}a_s(\boldsymbol{k},1)^{(m)}\delta_\Gamma(\boldsymbol{l},\boldsymbol{k'})\delta_{-r,s'}\delta_{n,p}(|\boldsymbol{l}\rangle\langle\boldsymbol{k}|\otimes a_r(\boldsymbol{l},1))^{(n)}a_{s'}(\boldsymbol{k'},1)^{(p)}\delta_\Gamma(\boldsymbol{k},\boldsymbol{l})\delta_{-r,s}\delta_{n,m}\Big) \\
= & -2\sum_{s,s'=\pm}e^{-2is\alpha}\int d\Gamma(\boldsymbol{k})\ \bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{k})a_{-s}(\boldsymbol{l},N)a_{s'}(\boldsymbol{k},N).
\end{aligned}
\tag{I.7}
$$

The following two formulas are calculated explicitly using (H.13), (I.6) and (I.7) for (569) and (570) in section 9.1

$$
\begin{aligned}
& [U(\Lambda,0,N)^\dagger Y_\alpha(\boldsymbol{l},N)U(\Lambda,0,N),\Psi(N)] = [Y_{\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})}(\boldsymbol{\Lambda^{-1}l},N),\Psi(N)] \\
= & \ 2\sum_{s,s'=\pm}e^{2is\alpha}e^{-4is\Theta(\Lambda,\boldsymbol{l})}\int d\Gamma(\boldsymbol{k})\ \psi_{ss'}(\boldsymbol{\Lambda^{-1}l},\boldsymbol{k})a_{-s}(\boldsymbol{\Lambda^{-1}l},N)^\dagger a_{s'}(\boldsymbol{k},N)^\dagger,
\end{aligned}
\tag{I.8}
$$

$$
\begin{aligned}
& [U(\Lambda,0,N)^\dagger Y_\alpha(\boldsymbol{l},N)U(\Lambda,0,N),\Psi(N)^\dagger] = [Y_{\alpha-2\Theta(\Lambda,\boldsymbol{\Lambda l})}(\boldsymbol{\Lambda^{-1}l},N),\Psi(N)^\dagger] \\
= & -2\sum_{s,s'=\pm}e^{-2is\alpha}e^{4is\Theta(\Lambda,\boldsymbol{l})}\int d\Gamma(\boldsymbol{k})\ \bar{\psi}_{ss'}(\boldsymbol{\Lambda^{-1}l},\boldsymbol{k})a_{-s}(\boldsymbol{\Lambda^{-1}l},N)a_{s'}(\boldsymbol{k},N).
\end{aligned}
\tag{I.9}
$$

J Explicit calculations of scalar products

$$
\begin{aligned}
& \langle O(N)|I(\boldsymbol{k},N)I(\boldsymbol{k}',N)|O(N)\rangle \\
= \; & \frac{1}{N^2}\sum_{m,n=1}^{N}\langle O(1)|^{\otimes N} I(\boldsymbol{k},1)^{(m)} I(\boldsymbol{k}',1)^{(n)}|O(1)\rangle^{\otimes N} \\
= \; & \frac{1}{N^2}\sum_{m=n=1}^{N}\langle O(1)|^{\otimes N} I(\boldsymbol{k},1)^{(m)} I(\boldsymbol{k}',1)^{(n)}|O(1)\rangle^{\otimes N} \\
+ \; & \frac{1}{N^2}\sum_{m\neq n=1}^{N}\langle O(1)|^{\otimes N} I(\boldsymbol{k},1)^{(m)} I(\boldsymbol{k}',1)^{(n)}|O(1)\rangle^{\otimes N} \\
= \; & \frac{1}{N}\langle O(1)|I(\boldsymbol{k},1)I(\boldsymbol{k}',1)|O(1)\rangle + \frac{N-1}{N}\langle O(1)|I(\boldsymbol{k},1)|O(1)\rangle\langle O(1)|I(\boldsymbol{k}',1)|O(1)\rangle \\
= \; & \frac{1}{N}\left(\bar{O}(\boldsymbol{k})O(\boldsymbol{k}')\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}') + (N-1)Z(\boldsymbol{k})Z(\boldsymbol{k}')\right). \qquad \text{(J.1)}
\end{aligned}
$$

The following formula is calculated explicitly, using commutation relation (I.3) and previous formula (J.1) for (450) in section 6.1

$$
\begin{aligned}
& \langle O(N)|\Psi(N)^\dagger\Psi(N)|O(N)\rangle \\
= \; & -\langle O(N)|[\Psi(N),\Psi(N)^\dagger]|O(N)\rangle \\
= \; & \langle O(N)|\sum_{rr'=\pm}\sum_{s,s'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')d\Gamma(\boldsymbol{l})d\Gamma(\boldsymbol{l}')\;\psi_{ss'}(\boldsymbol{k},\boldsymbol{k}')\bar{\psi}_{rr'}(\boldsymbol{l},\boldsymbol{l}') \\
\times \; & \left(a_r(\boldsymbol{l},N)a_{s'}(\boldsymbol{k}',N)^\dagger I(\boldsymbol{k},N)\delta_{s,r'}\delta_\Gamma(\boldsymbol{k},\boldsymbol{l}') + a_{r'}(\boldsymbol{l}',N)a_{s'}(\boldsymbol{k}',N)^\dagger I(\boldsymbol{k},N)\delta_{s,r}\delta_\Gamma(\boldsymbol{k},\boldsymbol{l})\right)|O(N)\rangle \\
= \; & \langle O(N)|\sum_{rr'=\pm}\sum_{s,s'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')d\Gamma(\boldsymbol{l})d\Gamma(\boldsymbol{l}')\;\psi_{ss'}(\boldsymbol{k},\boldsymbol{k}')\bar{\psi}_{rr'}(\boldsymbol{l},\boldsymbol{l}') \\
\times \; & (\delta_{s',r}\delta_\Gamma(\boldsymbol{k}',\boldsymbol{l})\delta_{s,r'}\delta_\Gamma(\boldsymbol{k},\boldsymbol{l}')I(\boldsymbol{k},N)I(\boldsymbol{k}',N) + \delta_{s',r'}\delta_\Gamma(\boldsymbol{k}',\boldsymbol{l}')\delta_{s,r}\delta_\Gamma(\boldsymbol{k},\boldsymbol{l})I(\boldsymbol{k},N)I(\boldsymbol{k}',N))|O(N)\rangle \\
= \; & \langle O(N)|\sum_{rr'=\pm}\sum_{s,s'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')d\Gamma(\boldsymbol{l})d\Gamma(\boldsymbol{l}')\;\delta_{s',r}\delta_\Gamma(\boldsymbol{k}',\boldsymbol{l})\delta_{s,r'}\delta_\Gamma(\boldsymbol{k},\boldsymbol{l}')I(\boldsymbol{k},N)I(\boldsymbol{k}',N)\psi_{ss'}(\boldsymbol{k},\boldsymbol{k}')\bar{\psi}_{rr'}(\boldsymbol{l},\boldsymbol{l}') \\
+ \; & \sum_{rr'=\pm}\sum_{s,s'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')d\Gamma(\boldsymbol{l})d\Gamma(\boldsymbol{l}')\;\delta_{s',r'}\delta_\Gamma(\boldsymbol{k}',\boldsymbol{l}')\delta_{s,r}\delta_\Gamma(\boldsymbol{k},\boldsymbol{l})I(\boldsymbol{k},N)I(\boldsymbol{k}',N)\psi_{ss'}(\boldsymbol{k},\boldsymbol{k}')\bar{\psi}_{rr'}(\boldsymbol{l},\boldsymbol{l}')|O(N)\rangle \\
= \; & \langle O(N)|\sum_{s,s'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\psi_{ss'}(\boldsymbol{k},\boldsymbol{k}')\bar{\psi}_{s's}(\boldsymbol{k}',\boldsymbol{k})I(\boldsymbol{k},N)I(\boldsymbol{k}',N) \\
+ \; & \sum_{s,s'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\psi_{ss'}(\boldsymbol{k},\boldsymbol{k}')\bar{\psi}_{ss'}(\boldsymbol{k},\boldsymbol{k}')I(\boldsymbol{k},N)I(\boldsymbol{k}',N)|O(N)\rangle \\
= \; & 2\langle O(N)|\sum_{s,s'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')I(\boldsymbol{k},N)I(\boldsymbol{k}',N)|\psi_{ss'}(\boldsymbol{k},\boldsymbol{k}')|^2|O(N)\rangle \\
= \; & 2\sum_{s,s'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|\psi_{ss'}(\boldsymbol{k},\boldsymbol{k}')|^2\langle O(N)|I(\boldsymbol{k},N)I(\boldsymbol{k}',N)|O(N)\rangle \\
= \; & \frac{2}{N}\sum_{s,s'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|\psi_{ss'}(\boldsymbol{k},\boldsymbol{k}')|^2\left(\bar{O}(\boldsymbol{k})O(\boldsymbol{k}')\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}') + (N-1)Z(\boldsymbol{k})Z(\boldsymbol{k}')\right) \\
= \; & \frac{2}{N}\sum_{s,s'=\pm}\int d\Gamma(\boldsymbol{k})|\psi_{ss'}(\boldsymbol{k},\boldsymbol{k})|^2 Z(\boldsymbol{k}) + \frac{2(N-1)}{N}\sum_{s,s'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')|\psi_{ss'}(\boldsymbol{k},\boldsymbol{k}')|^2 Z(\boldsymbol{k})Z(\boldsymbol{k}'). \qquad \text{(J.2)}
\end{aligned}
$$

The following formula is derived explicitly using (I.2) and (J.1) for (543) in section 8.1:

$$
\begin{aligned}
& \langle O(N)|\Psi(N)^\dagger Y_\beta(\boldsymbol{l},N)Y_\alpha(\boldsymbol{l}',N)\Psi(N)|O(N)\rangle \\
= \ & \langle O(N)|[\Psi(N)^\dagger, Y_\beta(\boldsymbol{l},N)][Y_\alpha(\boldsymbol{l}',N),\Psi(N)]|O(N)\rangle \\
= \ & 4\ \langle O(N)| \left(\sum_{s,s'=\pm} e^{-2is\beta}\int d\Gamma(\boldsymbol{k})\ \bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{k})a_{-s}(\boldsymbol{l},N)a_{s'}(\boldsymbol{k},N)\right) \\
\times \ & \left(\sum_{r,r'=\pm} e^{2ir\alpha}\int d\Gamma(\boldsymbol{k}')\ \psi_{rr'}(\boldsymbol{l}',\boldsymbol{k}')a_{-r}(\boldsymbol{l}',N)^\dagger a_{r'}(\boldsymbol{k}',N)^\dagger\right)|O(N)\rangle \\
= \ & 4\sum_{ss'rr'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{-2i(s\beta-r\alpha)}\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{k})\psi_{rr'}(\boldsymbol{l}',\boldsymbol{k}') \\
\times \ & \langle O(N)|a_{-s}(\boldsymbol{l},N)a_{s'}(\boldsymbol{k},N)a_{-r}(\boldsymbol{l}',N)^\dagger a_{r'}(\boldsymbol{k}',N)^\dagger|O(N)\rangle \\
= \ & 4\sum_{ss'rr'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{-2i(s\beta-r\alpha)}\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{k})\psi_{rr'}(\boldsymbol{l}',\boldsymbol{k}') \\
\times \ & \langle O(N)|[a_{-s}(\boldsymbol{l},N)a_{s'}(\boldsymbol{k},N), a_{-r}(\boldsymbol{l}',N)^\dagger a_{r'}(\boldsymbol{k}',N)^\dagger]|O(N)\rangle \\
= \ & 4\sum_{ss'rr'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{-2i(s\beta-r\alpha)}\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{k})\psi_{rr'}(\boldsymbol{l}',\boldsymbol{k}') \\
\times \ & \langle O(N)|\big(a_{-s}(\boldsymbol{l},N)a_{r'}(\boldsymbol{k}',N)^\dagger I(\boldsymbol{l}',N)\delta_{-r,s'}\delta_\Gamma(\boldsymbol{l}',\boldsymbol{k}) + a_{s'}(\boldsymbol{k},N)a_{r'}(\boldsymbol{k}',N)^\dagger I(\boldsymbol{l}',N)\delta_{-s,-r}\delta_\Gamma(\boldsymbol{l},\boldsymbol{l}')\big)|O(N)\rangle \\
= \ & 4\sum_{ss'rr'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{-2i(s\beta-r\alpha)}\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{k})\psi_{rr'}(\boldsymbol{l}',\boldsymbol{k}') \\
\times \ & \langle O(N)|\left(I(\boldsymbol{l},N)I(\boldsymbol{l}',N)\delta_\Gamma(\boldsymbol{l},\boldsymbol{k}')\delta_\Gamma(\boldsymbol{l}',\boldsymbol{k})\delta_{-s,r'}\delta_{-r,s'} + I(\boldsymbol{k}',N)I(\boldsymbol{l}',N)\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}')\delta_\Gamma(\boldsymbol{l},\boldsymbol{l}')\delta_{r',s'}\delta_{-s,-r}\right)|O(N)\rangle \\
= \ & 4\sum_{ss'rr'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{-2i(s\beta-r\alpha)}\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{k})\psi_{rr'}(\boldsymbol{l}',\boldsymbol{k}')\delta_\Gamma(\boldsymbol{l},\boldsymbol{k}')\delta_\Gamma(\boldsymbol{l}',\boldsymbol{k})\delta_{-s,r'}\delta_{-r,s'} \\
\times \ & \langle O(N)|I(\boldsymbol{l},N)I(\boldsymbol{l}',N)|O(N)\rangle \\
+ \ & 4\sum_{ss'rr'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{-2i(s\beta-r\alpha)}\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{k})\psi_{rr'}(\boldsymbol{l}',\boldsymbol{k}')\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}')\delta_\Gamma(\boldsymbol{l},\boldsymbol{l}')\delta_{r',s'}\delta_{-s,-r} \\
\times \ & \langle O(N)|I(\boldsymbol{k}',N)I(\boldsymbol{l}',N)|O(N)\rangle \\
= \ & 4\sum_{ss'=\pm} e^{-2i(s\beta+s'\alpha)}\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{l}')\psi_{-s'-s}(\boldsymbol{l}',\boldsymbol{l})\langle O(N)|I(\boldsymbol{l},N)I(\boldsymbol{l}',N)|O(N)\rangle \\
+ \ & 4\sum_{ss'=\pm}\int d\Gamma(\boldsymbol{k})\ e^{-2i(s\beta-s\alpha)}\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{k})\psi_{ss'}(\boldsymbol{l}',\boldsymbol{k})\delta_\Gamma(\boldsymbol{l},\boldsymbol{l}')\langle O(N)|I(\boldsymbol{k},N)I(\boldsymbol{l},N)|O(N)\rangle.
\end{aligned}
\tag{J.3}
$$

This formula is derived explicitly using (I.8), (I.9) and (J.1) for (571) in section 9.1:

$$
\begin{aligned}
& \langle O(N)|\Psi(N)^\dagger U(\Lambda,0,N)^\dagger Y_\beta(\boldsymbol{l},N)Y_\alpha(\boldsymbol{l}',N)U(\Lambda,0,N)\Psi(N)|O(N)\rangle \\
= \;& \langle O(N)|[\Psi(N)^\dagger, U(\Lambda,0,N)^\dagger Y_\beta(\boldsymbol{l},N)U(\Lambda,0,N)][U(\Lambda,0,N)^\dagger Y_\alpha(\boldsymbol{l}',N)U(\Lambda,0,N),\Psi(N)]|O(N)\rangle \\
= \;& 4\,\langle O(N)|\left(\sum_{s,s'=\pm} e^{-2is\beta}e^{4is\Theta(\Lambda,\boldsymbol{l})}\int d\Gamma(\boldsymbol{k})\;\bar{\psi}_{ss'}(\boldsymbol{\Lambda^{-1}l},\boldsymbol{k})a_{-s}(\boldsymbol{\Lambda^{-1}l},N)a_{s'}(\boldsymbol{k},N)\right) \\
\times \;& \left(\sum_{r,r'=\pm} e^{2ir\alpha}e^{-4ir\Theta(\Lambda,\boldsymbol{l}')}\int d\Gamma(\boldsymbol{k}')\;\psi_{rr'}(\boldsymbol{\Lambda^{-1}l'},\boldsymbol{k}')a_{-r}(\boldsymbol{\Lambda^{-1}l'},N)^\dagger a_{r'}(\boldsymbol{k}',N)^\dagger\right)|O(N)\rangle \\
= \;& 4\sum_{ss'rr'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\;e^{-2i(s\beta-r\alpha)}e^{4i(s\Theta(\Lambda,\boldsymbol{l})-r\Theta(\Lambda,\boldsymbol{l}'))}\bar{\psi}_{ss'}(\boldsymbol{\Lambda^{-1}l},\boldsymbol{k})\psi_{rr'}(\boldsymbol{\Lambda^{-1}l'},\boldsymbol{k}') \\
\times \;& \langle O(N)|a_{-s}(\boldsymbol{\Lambda^{-1}l},N)a_{s'}(\boldsymbol{k},N)a_{-r}(\boldsymbol{\Lambda^{-1}l'},N)^\dagger a_{r'}(\boldsymbol{k}',N)^\dagger|O(N)\rangle \\
= \;& 4\sum_{ss'rr'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\;e^{-2i(s\beta-r\alpha)}e^{4i(s\Theta(\Lambda,\boldsymbol{l})-r\Theta(\Lambda,\boldsymbol{l}'))}\bar{\psi}_{ss'}(\boldsymbol{\Lambda^{-1}l},\boldsymbol{k})\psi_{rr'}(\boldsymbol{\Lambda^{-1}l'},\boldsymbol{k}') \\
\times \;& \langle O(N)|[a_{-s}(\boldsymbol{\Lambda^{-1}l},N)a_{s'}(\boldsymbol{k},N),a_{-r}(\boldsymbol{\Lambda^{-1}l'},N)^\dagger a_{r'}(\boldsymbol{k}',N)^\dagger]|O(N)\rangle \\
= \;& 4\sum_{ss'rr'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\;e^{-2i(s\beta-r\alpha)}e^{4i(s\Theta(\Lambda,\boldsymbol{l})-r\Theta(\Lambda,\boldsymbol{l}'))}\bar{\psi}_{ss'}(\boldsymbol{\Lambda^{-1}l},\boldsymbol{k})\psi_{rr'}(\boldsymbol{\Lambda^{-1}l'},\boldsymbol{k}') \\
\times \;& \langle O(N)|(a_{-s}(\boldsymbol{\Lambda^{-1}l},N)a_{r'}(\boldsymbol{k}',N)^\dagger I(\boldsymbol{\Lambda^{-1}l'},N)\delta_{-r,s'}\delta_\Gamma(\boldsymbol{\Lambda^{-1}l'},\boldsymbol{k}) \\
+ \;& a_{s'}(\boldsymbol{k},N)a_{r'}(\boldsymbol{k}',N)^\dagger I(\boldsymbol{\Lambda^{-1}l'},N)\delta_{-s,-r}\delta_\Gamma(\boldsymbol{\Lambda^{-1}l},\boldsymbol{\Lambda^{-1}l'}))|O(N)\rangle \\
= \;& 4\sum_{ss'rr'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\;e^{-2i(s\beta-r\alpha)}e^{4i(s\Theta(\Lambda,\boldsymbol{l})-r\Theta(\Lambda,\boldsymbol{l}'))}\bar{\psi}_{ss'}(\boldsymbol{\Lambda^{-1}l},\boldsymbol{k})\psi_{rr'}(\boldsymbol{\Lambda^{-1}l'},\boldsymbol{k}') \\
\times \;& \langle O(N)|(I(\boldsymbol{\Lambda^{-1}l},N)I(\boldsymbol{\Lambda^{-1}l'},N)\delta_\Gamma(\boldsymbol{\Lambda^{-1}l},\boldsymbol{k}')\delta_\Gamma(\boldsymbol{\Lambda^{-1}l'},\boldsymbol{k})\delta_{-s,r'}\delta_{-r,s'} \\
+ \;& I(\boldsymbol{k}',N)I(\boldsymbol{\Lambda^{-1}l'},N)\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}')\delta_\Gamma(\boldsymbol{\Lambda^{-1}l},\boldsymbol{\Lambda^{-1}l'})\delta_{r',s'}\delta_{-s,-r})|O(N)\rangle \\
= \;& 4\sum_{ss'rr'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\;e^{-2i(s\beta-r\alpha)}e^{4i(s\Theta(\Lambda,\boldsymbol{l})-r\Theta(\Lambda,\boldsymbol{l}'))}\bar{\psi}_{ss'}(\boldsymbol{\Lambda^{-1}l},\boldsymbol{k})\psi_{rr'}(\boldsymbol{\Lambda^{-1}l'},\boldsymbol{k}') \\
\times \;& \langle O(N)|I(\boldsymbol{\Lambda^{-1}l},N)I(\boldsymbol{\Lambda^{-1}l'},N)|O(N)\rangle\delta_\Gamma(\boldsymbol{\Lambda^{-1}l},\boldsymbol{k}')\delta_\Gamma(\boldsymbol{\Lambda^{-1}l'},\boldsymbol{k})\delta_{-s,r'}\delta_{-r,s'} \\
+ \;& 4\sum_{ss'rr'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\;e^{-2i(s\beta-r\alpha)}e^{4i(s\Theta(\Lambda,\boldsymbol{l})-r\Theta(\Lambda,\boldsymbol{l}'))}\bar{\psi}_{ss'}(\boldsymbol{\Lambda^{-1}l},\boldsymbol{k})\psi_{rr'}(\boldsymbol{\Lambda^{-1}l'},\boldsymbol{k}') \\
\times \;& \langle O(N)|I(\boldsymbol{k}',N)I(\boldsymbol{\Lambda^{-1}l'},N)|O(N)\rangle\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}')\delta_\Gamma(\boldsymbol{\Lambda^{-1}l},\boldsymbol{\Lambda^{-1}l'})\delta_{r',s'}\delta_{-s,-r} \\
= \;& 4\sum_{ss'=\pm} e^{-2i(s\beta+s'\alpha)}e^{4i(s\Theta(\Lambda,\boldsymbol{l})+s'\Theta(\Lambda,\boldsymbol{l}'))}\bar{\psi}_{ss'}(\boldsymbol{\Lambda^{-1}l},\boldsymbol{\Lambda^{-1}l'})\psi_{-s'-s}(\boldsymbol{\Lambda^{-1}l'},\boldsymbol{\Lambda^{-1}l}) \\
\times \;& \langle O(N)|I(\boldsymbol{\Lambda^{-1}l},N)I(\boldsymbol{\Lambda^{-1}l'},N)|O(N)\rangle \\
+ \;& 4\sum_{ss'=\pm}\int d\Gamma(\boldsymbol{k})\;e^{-2i(s\beta-s\alpha)}e^{4i(s\Theta(\Lambda,\boldsymbol{l})-s\Theta(\Lambda,\boldsymbol{l}'))}\bar{\psi}_{ss'}(\boldsymbol{\Lambda^{-1}l},\boldsymbol{k})\psi_{ss'}(\boldsymbol{\Lambda^{-1}l'},\boldsymbol{k})\delta_\Gamma(\boldsymbol{\Lambda^{-1}l},\boldsymbol{\Lambda^{-1}l'}) \\
\times \;& \langle O(N)|I(\boldsymbol{k},N)I(\boldsymbol{\Lambda^{-1}l},N)|O(N)\rangle. \qquad \text{(J.4)}
\end{aligned}
$$

This formula is derived explicitly using (I.8), (I.9) and (J.1) for (589) in section 9.4:

$$
\begin{aligned}
& \langle O(N)|\Psi(N)^\dagger Y_\beta(\boldsymbol{l},N)U(\Lambda,0,N)^\dagger Y_\alpha(\boldsymbol{l}',N)U(\Lambda,0,N)\Psi(N)|O(N)\rangle \\
= & \ \langle O(N)|[\Psi(N)^\dagger, Y_\beta(\boldsymbol{l},N)][U(\Lambda,0,N)^\dagger Y_\alpha(\boldsymbol{l}',N)U(\Lambda,0,N),\Psi(N)]|O(N)\rangle \\
= & \ 4\,\langle O(N)|\left(\sum_{s,s'=\pm} e^{-2is\beta}\int d\Gamma(\boldsymbol{k})\ \bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{k})a_{-s}(\boldsymbol{l},N)a_{s'}(\boldsymbol{k},N)\right) \\
\times & \ \left(\sum_{r,r'=\pm} e^{2ir\alpha}e^{-4ir\Theta(\Lambda,\boldsymbol{l}')}\int d\Gamma(\boldsymbol{k}')\ \psi_{rr'}(\boldsymbol{\Lambda^{-1}l}',\boldsymbol{k}')a_{-r}(\boldsymbol{\Lambda^{-1}l}',N)^\dagger a_{r'}(\boldsymbol{k}',N)^\dagger\right)|O(N)\rangle \\
= & \ 4\sum_{ss'rr'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{-2i(s\beta-r\alpha)}e^{-4ir\Theta(\Lambda,\boldsymbol{l}')}\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{k})\psi_{rr'}(\boldsymbol{\Lambda^{-1}l}',\boldsymbol{k}') \\
\times & \ \langle O(N)|a_{-s}(\boldsymbol{l},N)a_{s'}(\boldsymbol{k},N)a_{-r}(\boldsymbol{\Lambda^{-1}l}',N)^\dagger a_{r'}(\boldsymbol{k}',N)^\dagger|O(N)\rangle \\
= & \ 4\sum_{ss'rr'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{-2i(s\beta-r\alpha)}e^{-4ir\Theta(\Lambda,\boldsymbol{l}')}\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{k})\psi_{rr'}(\boldsymbol{\Lambda^{-1}l}',\boldsymbol{k}') \\
\times & \ \langle O(N)|[a_{-s}(\boldsymbol{l},N)a_{s'}(\boldsymbol{k},N),a_{-r}(\boldsymbol{\Lambda^{-1}l}',N)^\dagger a_{r'}(\boldsymbol{k}',N)^\dagger]|O(N)\rangle \\
= & \ 4\sum_{ss'rr'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{-2i(s\beta-r\alpha)}e^{-4ir\Theta(\Lambda,\boldsymbol{l}')}\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{k})\psi_{rr'}(\boldsymbol{\Lambda^{-1}l}',\boldsymbol{k}') \\
\times & \ \langle O(N)|\big(a_{-s}(\boldsymbol{l},N)a_{r'}(\boldsymbol{k}',N)^\dagger I(\boldsymbol{\Lambda^{-1}l}',N)\delta_{-r,s'}\delta_\Gamma(\boldsymbol{\Lambda^{-1}l}',\boldsymbol{k}) \\
+ & \ a_{s'}(\boldsymbol{k},N)a_{r'}(\boldsymbol{k}',N)^\dagger I(\boldsymbol{\Lambda^{-1}l}',N)\delta_{-s,-r}\delta_\Gamma(\boldsymbol{l},\boldsymbol{\Lambda^{-1}l}')\big)|O(N)\rangle \\
= & \ 4\sum_{ss'rr'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{-2i(s\beta-r\alpha)}e^{-4ir\Theta(\Lambda,\boldsymbol{l}')}\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{k})\psi_{rr'}(\boldsymbol{\Lambda^{-1}l}',\boldsymbol{k}') \\
\times & \ \langle O(N)|\big(I(\boldsymbol{l},N)I(\boldsymbol{\Lambda^{-1}l}',N)\delta_\Gamma(\boldsymbol{l},\boldsymbol{k}')\delta_\Gamma(\boldsymbol{l}',\boldsymbol{k})\delta_{-s,r'}\delta_{-r,s'} \\
+ & \ I(\boldsymbol{k}',N)I(\boldsymbol{\Lambda^{-1}l}',N)\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}')\delta_\Gamma(\boldsymbol{l},\boldsymbol{\Lambda^{-1}l}')\delta_{r',s'}\delta_{-s,-r}\big)|O(N)\rangle \\
= & \ 4\sum_{ss'rr'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{-2i(s\beta-r\alpha)}e^{-4ir\Theta(\Lambda,\boldsymbol{l}')}\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{k})\psi_{rr'}(\boldsymbol{\Lambda^{-1}l}',\boldsymbol{k}') \\
\times & \ \langle O(N)|I(\boldsymbol{l},N)I(\boldsymbol{\Lambda^{-1}l}',N)|O(N)\rangle\delta_\Gamma(\boldsymbol{l},\boldsymbol{k}')\delta_\Gamma(\boldsymbol{\Lambda^{-1}l}',\boldsymbol{k})\delta_{-s,r'}\delta_{-r,s'}|O(N)\rangle \\
+ & \ 4\sum_{ss'rr'=\pm}\int d\Gamma(\boldsymbol{k})d\Gamma(\boldsymbol{k}')\ e^{-2i(s\beta-r\alpha)}e^{-4ir\Theta(\Lambda,\boldsymbol{l}')}\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{k})\psi_{rr'}(\boldsymbol{\Lambda^{-1}l}',\boldsymbol{k}') \\
\times & \ \langle O(N)|I(\boldsymbol{k}',N)I(\boldsymbol{\Lambda^{-1}l}',N)|O(N)\rangle\delta_\Gamma(\boldsymbol{k},\boldsymbol{k}')\delta_\Gamma(\boldsymbol{l},\boldsymbol{\Lambda^{-1}l}')\delta_{r',s'}\delta_{-s,-r}|O(N)\rangle \\
= & \ 4\sum_{ss'=\pm} e^{-2i(s\beta+s'\alpha)}e^{4is'\Theta(\Lambda,\boldsymbol{l}')}\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{\Lambda^{-1}l}')\psi_{-s'-s}(\boldsymbol{\Lambda^{-1}l}',\boldsymbol{l})\langle O(N)|I(\boldsymbol{l},N)I(\boldsymbol{\Lambda^{-1}l}',N)|O(N)\rangle \\
+ & \ 4\sum_{ss'=\pm}\int d\Gamma(\boldsymbol{k})\ e^{-2i(s\beta-s\alpha)}e^{4is\Theta(\Lambda,\boldsymbol{l}')}\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{k})\psi_{ss'}(\boldsymbol{\Lambda^{-1}l}',\boldsymbol{k})\delta_\Gamma(\boldsymbol{l},\boldsymbol{\Lambda^{-1}l}')\langle O(N)|I(\boldsymbol{k},N)I(\boldsymbol{l},N)|O(N)\rangle.
\end{aligned}
\tag{J.5}
$$

K Calculations for the EPR average

The following calculations are done to show formula (545) in section 8.2:

$$
\begin{aligned}
& \sum_{ss'=\pm} e^{-2i(s\beta+s'\alpha)} \bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{l}')\psi_{-s-s'}(\boldsymbol{l},\boldsymbol{l}') \\
= \quad & \sum_{ss'=\pm} e^{2i(s\beta+s'\alpha)} \bar{\psi}_{-s-s'}(\boldsymbol{l},\boldsymbol{l}')\psi_{ss'}(\boldsymbol{l},\boldsymbol{l}') \\
= \quad & \overline{\sum_{ss'=\pm} e^{-2i(s\beta+s'\alpha)} \bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{l}')\psi_{-s-s'}(\boldsymbol{l},\boldsymbol{l}')} \\
= \quad & \Re\left(\sum_{ss'=\pm} e^{-2i(s\beta+s'\alpha)} \bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{l}')\psi_{-s-s'}(\boldsymbol{l},\boldsymbol{l}')\right) \\
= \quad & \sum_{ss'=\pm} \left(\Re e^{-2i(s\beta+s'\alpha)} \Re\left(\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{l}')\psi_{-s-s'}(\boldsymbol{l},\boldsymbol{l}')\right) - \Im e^{-2i(s\beta+s'\alpha)} \Im\left(\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{l}')\psi_{-s-s'}(\boldsymbol{l},\boldsymbol{l}')\right)\right) \\
= \quad & \sum_{ss'=\pm} \left(\cos\left(2(s\beta+s'\alpha)\right) \Re\left(\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{l}')\psi_{-s-s'}(\boldsymbol{l},\boldsymbol{l}')\right) + \sin\left(2(s\beta+s'\alpha)\right) \Im\left(\bar{\psi}_{ss'}(\boldsymbol{l},\boldsymbol{l}')\psi_{-s-s'}(\boldsymbol{l},\boldsymbol{l}')\right)\right) \\
= \quad & \sum_{s=\pm} \left(\cos\left(2s(\beta+\alpha)\right) \Re\left(\bar{\psi}_{ss}(\boldsymbol{l},\boldsymbol{l}')\psi_{-s-s}(\boldsymbol{l},\boldsymbol{l}')\right) + \sin\left(2s(\beta+\alpha)\right) \Im\left(\bar{\psi}_{ss}(\boldsymbol{l},\boldsymbol{l}')\psi_{-s-s}(\boldsymbol{l},\boldsymbol{l}')\right)\right) \\
+ \quad & \sum_{s=\pm} \left(\cos\left(2s(\beta-\alpha)\right) \Re\left(\bar{\psi}_{s,-s}(\boldsymbol{l},\boldsymbol{l}')\psi_{-s,s}(\boldsymbol{l},\boldsymbol{l}')\right) + \sin\left(2s(\beta-\alpha)\right) \Im\left(\bar{\psi}_{s,-s}(\boldsymbol{l},\boldsymbol{l}')\psi_{-s,s}(\boldsymbol{l},\boldsymbol{l}')\right)\right) \\
= \quad & \cos 2(\beta+\alpha)\left(\Re\left(\bar{\psi}_{++}(\boldsymbol{l},\boldsymbol{l}')\psi_{--}(\boldsymbol{l},\boldsymbol{l}')\right) + \Re\left(\bar{\psi}_{--}(\boldsymbol{l},\boldsymbol{l}')\psi_{++}(\boldsymbol{l},\boldsymbol{l}')\right)\right) \\
+ \quad & \sin 2(\beta+\alpha)\left(\Im\left(\bar{\psi}_{++}(\boldsymbol{l},\boldsymbol{l}')\psi_{--}(\boldsymbol{l},\boldsymbol{l}')\right) - \Im\left(\bar{\psi}_{--}(\boldsymbol{l},\boldsymbol{l}')\psi_{++}(\boldsymbol{l},\boldsymbol{l}')\right)\right) \\
+ \quad & \cos 2(\beta-\alpha)\left(\Re\left(\bar{\psi}_{+-}(\boldsymbol{l},\boldsymbol{l}')\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')\right) + \Re\left(\bar{\psi}_{-+}(\boldsymbol{l},\boldsymbol{l}')\psi_{+-}(\boldsymbol{l},\boldsymbol{l}')\right)\right) \\
+ \quad & \sin 2(\beta-\alpha)\left(\Im\left(\bar{\psi}_{+-}(\boldsymbol{l},\boldsymbol{l}')\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')\right) - \Im\left(\bar{\psi}_{-+}(\boldsymbol{l},\boldsymbol{l}')\psi_{+-}(\boldsymbol{l},\boldsymbol{l}')\right)\right) \\
= \quad & 2\cos 2(\beta+\alpha)\Re\left(\bar{\psi}_{++}(\boldsymbol{l},\boldsymbol{l}')\psi_{--}(\boldsymbol{l},\boldsymbol{l}')\right) + 2\sin 2(\beta+\alpha)\Im\left(\bar{\psi}_{++}(\boldsymbol{l},\boldsymbol{l}')\psi_{--}(\boldsymbol{l},\boldsymbol{l}')\right) \\
+ \quad & 2\cos 2(\beta-\alpha)\Re\left(\bar{\psi}_{+-}(\boldsymbol{l},\boldsymbol{l}')\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')\right) + 2\sin 2(\beta-\alpha)\Im\left(\bar{\psi}_{+-}(\boldsymbol{l},\boldsymbol{l}')\psi_{-+}(\boldsymbol{l},\boldsymbol{l}')\right). \qquad \text{(K.1)}
\end{aligned}
$$

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