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English for MATHEMATICS

for students of technical studies

GDAŃSK 2015

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Introduction

"English for Mathematics" is a course book for learning English. The idea was to create a teaching aid which would enable students of technical faculties to become familiar with basic mathematical concepts in English.

The main aim of "English for Mathematics" is to be used during foreign language courses at university – it is not intended to be used for teaching math. However, it may be very helpful for students intending to take part in scholarships abroad. It can also be used by those interested in mathematical terminology in English.

"English for Mathematics" may be successfully used by teachers of English who were not trained in mathematics – it includes teacher's book with answers to all exercises, instructions and additional ideas. Each chapter includes phonetic exercises which should be helpful as far as pronunciation is concerned. There are also exercises with recorded mathematical texts, which give additional practice in listening and correct pronunciation. The recordings (in MP3 format) can be downloaded from CJO PG website and from the Mathematics portal (**http://www.pg.edu.pl/math**), **http://cjo.pg.edu.pl/** free of charge.

All the chapters can be used independently but their chronology is connected with the history of mathematics. We recommend the use of Appendices and Additional Material (including biographies of selected famous mathematicians). You can find there additional exercises, mathematical symbols and ways of reading mathematical operations. We also recommend encouraging students to use some up-to-date articles to prepare presentations towards the end of the course.

Acknowledgements

We would like to express special thanks to dr Anita Dąbrowicz-Tlałka for the enormous amount of time she spent on reading and discussing our work. We would also like to express our thanks to prof. Jerzy Topp for his advice at the early stages of the project. Very special thanks go to Łukasz Maciejewski for his professional support, advice, correction and unfailing patience, as well as to Marek Raczunas for all the support and advice he gave us. Last but not least, we would like to thank our colleague and friend, Lara Kalenik, who lent her beautiful voice for our recording.

Remark

There are various ways of noting multiplication throughout the book (· * ×) and different ways of writing decimal fractions, for example 0.75 or .75.

Chapter 1

☺**It is only two weeks into the term that, in a calculus class, a student raises his hand and asks: "*Will we ever need this stuff in real life?*" The professor gently smiles at him and says: "*Of course not – if your real life consists of flipping hamburgersat MacDonald's!*"**☺

Introduction

Lead-in:

1. Discuss the definition of mathematics in pairs.

2. Listen to the recording and repeat the following terms:

quantity, science, branch, space, equation, calculus, integer, topic, to solve, pattern, application.

3. Discuss and explain the meaning of these terms in pairs.

4. Read the text. Find any other new words.

Text:

What is mathematics, anyway?

We keep the *broad* definition here, that mathematics includes all the related areas which touch on quantitative, geometric, and logical themes. This includes Statistics, Computer Science, Logic, Applied Mathematics, and other fields which are frequently considered distinct from mathematics, as well as fields which study the study of mathematics – History of Mathematics, Mathematics Education, and so on. We draw the line only at experimental sciences, philosophy, and computer applications. Personal perspectives vary widely, of course.

A fairly standard definition is the one in the Columbia Encyclopedia (5^{th} ed.): "Mathematics: deductive study of numbers, geometry, and various abstract constructs, or structures. The latter often arise from analytical models in the empirical sciences, but may emerge from purely mathematical considerations."

Some definitions of mathematics heard from others are:

- That which mathematicians do.
- The study of well-defined things.
- The study of statements of the form "P implies Q".
- The science one could go on practicing should one wake up one morning to find that the world has ceased to exist. (attrib. to Bertrand Russell).

- The science of patterns (Keith Devlin).
- "Mathematics, at the beginning, is sometimes described as the science of Number and Space – better, of Number, Time, Space and Motion." – Saunders MacLane, in *Mathematics*: *Form and Function*.

Contrary to common perception, mathematics does not consist of "crunching numbers" or "solving equations". As we shall see there are branches of mathematics concerned with *setting up* equations, or *analyzing* their solutions, and there are parts of mathematics devoted to *creating methods* for doing computations. But there are also parts of mathematics which have nothing at all to do with numbers or equations.

Mathematics is often defined as the study of topics such as quantity, structure, space, and change. Another view, held by many mathematicians, is that mathematics is the body of knowledge justified by deductive reasoning, starting from axioms and definitions.

Mathematics arises wherever there are difficult problems that involve quantity, structure, space or change. At first these were found in commerce, land measurement and later astronomy. Nowadays, all fields of science suggest problems studied by mathematicians, and many problems arise within mathematics itself. Newton invented infinitesimal calculus and Feynman his Feynman path integral using a combination of reasoning and physical insight, and today's string theory also inspires new mathematics. Some mathematics is only relevant in the area that inspired it, and is applied to solve further problems in that area. But often mathematics inspired by one area proves useful in many areas, and joins the general stock of mathematical concepts. The remarkable fact that even the "purest" mathematics often turns out to have practical applications is what Eugene Wigner has called "the unreasonable effectiveness of mathematics".

Major themes in mathematics

An alphabetical and subclassified list of mathematics articles is available. The following list of themes and links gives just one possible view. For a fuller treatment, see areas of mathematics or the list of mathematics lists.

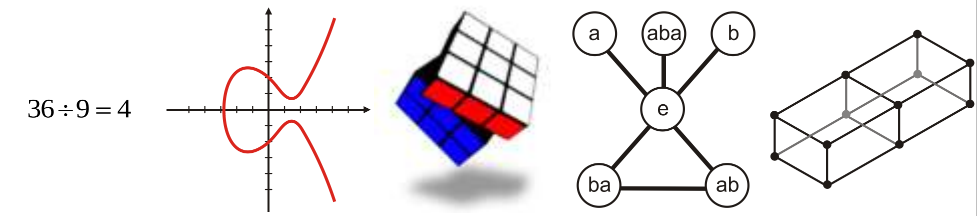
Quantity

Quantity starts with counting and measurement.

1, 2, ...	..., –1, 0, 1,...	$\frac{1}{2}$, $\frac{2}{3}$, 0.125, ...	π, e, $\sqrt{2}$, ...	i, $3i+2$, $e^{i\pi/3}$, ...
Natural numbers	Integers	Rational numbers	Irrational numbers	Complex numbers

Structure

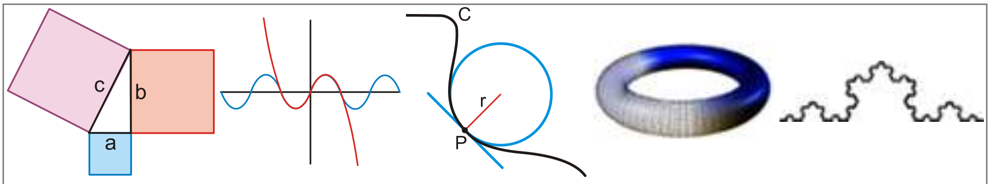
Pinning down ideas of size, symmetry, and mathematical structure.



Arithmetic | Number theory | Abstract algebra | Group theory | Order theory

Space

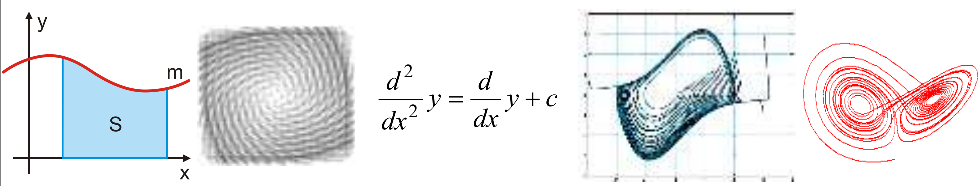
A more visual approach to mathematics.



Geometry | Trigonometry | Differential geometry | Topology | Fractal geometry

Change

Ways to express and handle change in mathematical functions, and changes between numbers.



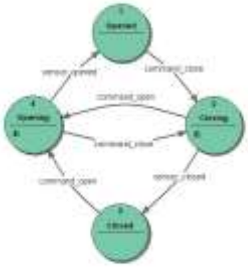
Calculus | Vector calculus | Differential equations | Dynamical systems | Chaos theory

Discrete mathematics

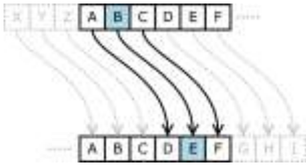
Discrete mathematics involves techniques that apply to objects that can only take on specific, separated values.

[1,2,3][1,3,2]
[2,1,3][2,3,1]
[3,1,2][3,2,1]

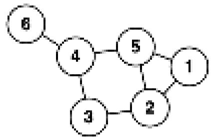
Combinatorics



Theory of computation



Cryptography



Graph theory

Adapted from www.math.niu.edu and Wikipedia

Follow-up exercises

5. Cover the text. Which fields of mathematics can you remember? Make a list and compare your results in pairs.

6. Find the mathematical terms for the following definitions:

a) an amount of something that can be counted or measured;
b) an arrangement of different parts of something in a pattern or system in which each part is connected to the other;
c) the part of mathematics that deals with e.g. changing quantities such as the speed of the falling stone or the shape of a curved line;
d) a quantity that has a direction as well as a size, usually represented as an arrow;
e) a statement in mathematics showing that two quantities are equal;
f) study of angles and shapes formed by a relationships of lines, surfaces and solid objects in space;
g) a word or sign which represents an amount or quantity;
h) a quantity that usually changes accordingly to how another mathematical quantity changes;
i) a division or a part of a whole number in mathematics.

7. Work in groups of four. One of you should say the definition, the others should give the correct word. Swap roles.

8. What are the plural forms of the following words?

quantity	branch
calculus	area
science	axiom
article	computation
perspective	consideration

What are the rules for forming plural nouns? Divide the nouns into groups according to the rule they follow.

9. Try to read the following operations.

1. The second(-order) derivative of function *y* of argument *x* equals the derivative of function *y* of argument *x* plus a constant

$$\frac{d^2}{dx^2}y = \frac{d}{dx}y + c$$

2. The scalar product of vectors with the following coordinates: 1, 2, 3 and 1, 3, 2
   [1,2,3]◦[1,3,2]

3. 5 minus 9 equals 5 plus minus 9, which equals minus 4
   $5 - 9 = 5 + (-9) = -4$.

4. 20 over 5 equals 4, minus 20 over 5 equals minus 4

$$\frac{20}{5} = 4, \ \frac{-20}{5} = -4 .$$

5. 14 divided by 2 equals 7, 14 divided by minus 2 equals minus 7
   $14 \div 2 = 7$, $14 \div (-2) = -7$,

6. Minus 1 times minus 2 times minus 1
   $(-1)(-2)(-1)$

7. Minus 3 times, open brackets, *x* plus 4, close brackets
   $-3(x + 4)$.

8. Minus 3 times, open brackets, *x* plus 4, close brackets, equals minus 3 *x* minus 3 times 4, which equals minus 3 *x* minus 12
   $-3(x + 4) = -3 \cdot x - 3 \cdot 4 = -3x - 12$.

9. Minus 3 all squared
   $(-3)^{2.}$

10. Minus 3 all squared equals minus 3 times minus 3, which equals (positive/plus) 3 times (positive/plus) 3, which equals 9
    $(-3)^2 = (-3)(-3) = 3 \cdot 3 = 9$.

11. Minus 3 all cubed
    $(-3)^3$

12. The square root of 16 equals 4

$$\sqrt{16} = 4$$

---

13. The cube root of minus 8 equals minus 2

$$\sqrt[3]{-8} = -2$$

---

14. 25 times, open brackets, 1 over *a*, close brackets, to the power of minus *n* times, open brackets, 2 *n* close brackets, to the power of 0 times 5 to (the power of) minus 3 times, open brackets, *a* over *x*, close brackets, to the power of minus *n* equals 5 to the power of 2 minus 3 times *a* to (the power of) *n* times *a* to (the power of) minus *n* times *x* to (the power of) minus minus *n* equals 5 to (the power of) minus 1 times *a* to (the power of) 0 times *x* to (the power of) *n* equals *x* to (the power of) *n* over 5.

$$25\cdot\left(\frac{1}{a}\right)^{-n}\cdot(2n)^0\cdot 5^{-3}\cdot\left(\frac{a}{x}\right)^{-n} = 5^{2-3}\cdot a^n\cdot a^{-n}\cdot x^{-(-n)} = 5^{-1}\cdot a^0\cdot x^n = \frac{x^n}{5}$$

---

15. 27 *a* to (the power of) 4 times *b* to (the power of) 4 times 56 *a* squared times *b* to (the power of) minus 3 times 42 *a* to (the power of) minus 2 times *b* cubed equals 3 cubed times *a* to (the power of) 4 times *b* to (the power of) 4 equals open brackets 2 squared times 3 squared times 7 *a* squared times *b* squared close brackets, squared.

$$27a^4b^4\cdot 56a^2b^{-3}\cdot 42a^{-2}b^3$$
$$= 3^3\cdot a^4b^4\cdot 7\cdot 2^3\cdot a^2b^{-3}\cdot 7\cdot 2\cdot 3\cdot a^{-2}b^3 = 2^4\cdot 3^4\cdot 7^2a^4b^4 = (2^2\cdot 3^2\cdot 7a^2b^2)^2$$

---

16. 2 times 10 (to the power of) minus 3 times, open brackets, 3 times 10 to (the power of) 10, close brackets, squared over 3 point 6 times 10 to (the power of) 13, kilo watt hours equals ….

$$\frac{2\cdot 10^{-3}\cdot(3\cdot 10^{10})^2}{3.6\cdot 10^{13}}\text{ kWh} = \frac{2\cdot 9\cdot 10^{-3}\cdot 10^{20}}{3.6\cdot 10^{13}}\text{ kWh} = 5\cdot 10^4\text{ kWh.}$$

---

10. Choose a biography of a famous mathematician form the “Additional material” at the end of the book. Read it, explain the unknown words. Prepare the most important facts and issues.

11. Work in groups of three. Each of you should have prepared a different biography. Tell your partners about “your” mathematician and his/her greatest achievements.

☺Teacher: "*Who can tell me what 7 times 6 is?*"
Student: "*It's 42!*"
Teacher: "*Very good! – And who can tell me what 6 times 7 is?*"
Same student: "*It's 24!*"☺

Algebra

Lead-in:

1. Which basic mathematical operations do you know? Discuss the issue in pairs.

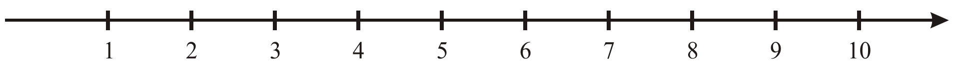
2. Listen to your teacher.

3. Listen and repeat:
exponent, *fraction*, *square*, *root*, *count*, *arrow*, *multiply*, *multiplication*, *divide*, *division*, *add*, *addition*, *subtract*, *subtraction*, *simplify*, *cancel*, *parentheses*, *decimal*.

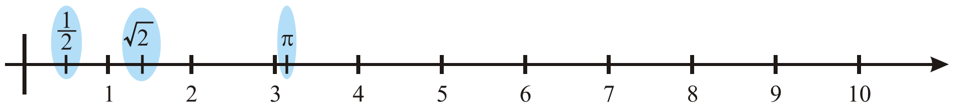
4. Read the text and find more useful mathematical terms.

Text:

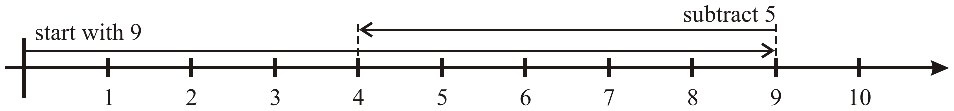
When you first learned your numbers, way back in elementary school, you learned the counting numbers: 1, 2, 3, 4, 5, 6, and so on. Your number line looked like this:



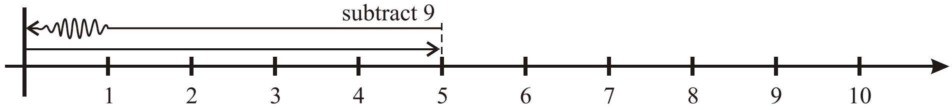
Later on, you learned about zero, fractions, decimals, square roots, and other types of numbers, so your number line started looking somewhat like this:



Addition, multiplication, and division always made sense – as long as you didn't try to divide by zero but sometimes subtraction didn't work. If you had "9 – 5", you got 4:



But what if you had "5 – 9"? You just couldn't do this, because there wasn't enough "space":



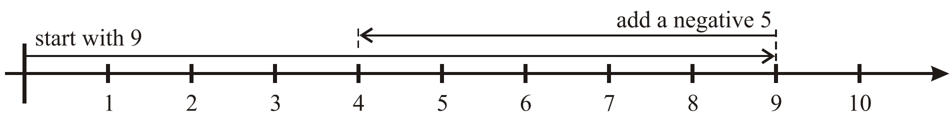
You can solve this "space" problem by using negative numbers. The "whole" numbers start at zero and count off to the right. The negatives start at zero and count off to the left:

Note the arrowhead on the far right end of the number line. The arrow tells you the direction in which the numbers are getting bigger. Then the arrow also tells you that the negatives are getting smaller as they move off to the left. That is, –5 is smaller than –4. This might seem a bit weird at first, but that's okay; negatives take some getting used to. Let's look at a few inequalities, to practice your understanding. Refer to the number line above, as necessary.

greater than or less than? 3 _____ 6 answer: 3 < 6
greater than or less than? –3 _____ 6 answer: –3 < 6
greater than or less than? –3 _____ –6 answer: –3 > –6
greater than or less than? 0 _____ 1 answer: 0 < 1
greater than or less than? 0 _____ –1 answer: 0 > –1

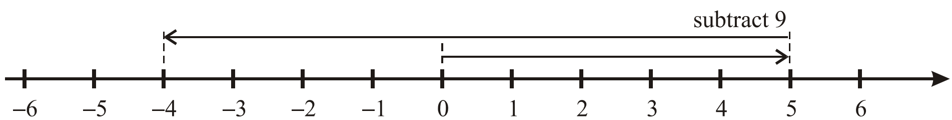
Addition and subtraction

How do you deal with adding and subtracting negatives? It works similarly to adding and subtracting positive numbers. If you are adding a negative, this is pretty much the same as subtracting a positive, if you view "adding a negative" as adding to the left. Let's return to the first example from above: "9 – 5" can also be written as "9 + (–5)". Graphically, it would be drawn as "an arrow from zero to nine, and then a 'negative' arrow five units long":

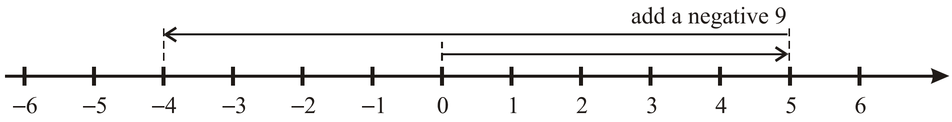


… and you get "9 + (–5) = 4".

Now look back at that subtraction you couldn't do: 5 – 9. Because you now have negatives off to the left of zero, you have the "space" to complete this subtraction. View the subtraction as adding a negative 9, that is, draw an arrow from zero to five, and then a "negative" arrow nine units long:



… or, which is the same thing:



Then 5 – 9 = 5 + (–9) = –4.

Of course, this method of counting off your answer on a number line won't work so well if you're dealing with larger numbers. For instance, think about doing "465 – 739". You certainly don't want to use a number line for this. You know that the answer to "465 – 739" has to be negative, (because "minus 739" will take you somewhere to the left of zero), but how do you figure out *which* negative number is the answer?

Look again at "5 – 9". You know now that the answer will be negative, because you're subtracting a bigger number than you started with (nine is bigger than five). The easiest way of dealing with this is to do the subtraction "normally" (with the smaller number subtracted from the larger number), and then put a "minus" sign in the answer: 9 – 5 = 4, so 5 – 9 = –4. This works the same way (and is simpler) for bigger numbers: 739 – 465 = 274, so 465 – 739 = –274.

Adding two negative numbers is easy: you're just adding two "negative" arrows, so it's just the backwards of "regular" addition. For instance, 4 + 6 = 10, and –4 – 6 = –4 + (–6) = –10. But what about when you have lots of both positive and negative numbers? For instance:

Simplify 18 – (–16) – 3 – (–5) + 2.

Probably the simplest thing to do is convert everything to addition, group the positives together and the negatives together, combine, and simplify. It looks like this:

$$\begin{aligned} &18 - (-16) - 3 - (-5) + 2 \\ &\quad = 18 + 16 - 3 + 5 + 2 \\ &\quad = 18 + 16 + 5 + 2 - 3 \\ &\quad = 41 - 3 \\ &\quad = 38. \end{aligned}$$

Multiplication and division

Turning from addition and subtraction, how do you do multiplication and division with negatives? Actually, we've already covered the hard part: you already know the "sign" rules:

plus times plus is plus (adding many hot cubes raises the temperature)

minus times plus is minus (removing many hot cubes reduces the temperature)

plus times minus is minus (adding many cold cubes reduces the temperature)

minus times minus is plus (removing many cold cubes raises the temperature)

The sign rules work the same way for division; just replace "times" with "divided by". Here are a couple examples of the rules in division:

$$\frac{20}{5} = 4, \quad \frac{-20}{5} = -4, \quad \frac{20}{-5} = -4, \quad \frac{-20}{-5} = 4.$$

(Remember that fractions are just another form of division!)

$$14 \div 2 = 7, \;\; 14 \div (-2) = -7, \;\; (-14) \div 2 = -7, \;\; (-14) \div (-2) = 7.$$

You may notice people "canceling off" minus signs. They are taking advantage of the fact that "minus times minus is plus". For instance, suppose you have (–2)(–3)(–4). Any two negatives, when multiplied together, become one positive. So pick any two of the multiplied (or divided) negatives, and "cancel" their signs:

Simplify (–2)(–3)(–4)

$(-2)(-3)(-4)$
$= 6(-4)$
$= -24.$

So if you're given a long multiplication with negatives, just cancel off in pairs:

Simplify (–1)(–2)(–1)(–3)(–4)(–2)(–1)

$(-1)(-2)(-1)(-3)(-4)(-2)(-1)$
$= 1\cdot 2\cdot 1\cdot 3\cdot(-4)(-2)(-1)$
$= 1\cdot 2\cdot 1\cdot 3\cdot(+4)(+2)(-1)$
$= 1\cdot 2\cdot 1\cdot 3\cdot 4\cdot 2\cdot(-1)$
$= 2\cdot 3\cdot 4\cdot 2\cdot(-1)$
$= 48\cdot(-1)$
$= -48.$

Negatives through parentheses

The major difficulty that people have with negatives is in dealing with parentheses; particularly, in taking a negative through parentheses. The usual situation is something like this:

$$-3(x + 4).$$

If you had "$3(x + 4)$", you would know to "distribute" the 3 "over" the parentheses:

$$3(x + 4) = 3x + 3\cdot 4 = 3x + 12.$$

The same rules apply when you're dealing with negatives. If you have trouble keeping track, use little arrows:

$$-3(x + 4) = -3(x + 4) = -3x - 3\cdot 4 = -3x - 12.$$

Here are a couple more examples:

Simplify 3(*x* – 5)

$3(x - 5) = 3x + 3\cdot(-5) = 3x - 15.$

Simplify –2(*x* – 3)

$-2(x - 3) = -2x - 2\cdot(-3) = -2x + 2\cdot 3 = -2x + 6.$

Exponents

Now you can move on to exponents, using the cancellation-of-minus-signs property of multiplication. For instance, $(3)^2 = 3 \cdot 3 = 9$. In the same way:

Simplify $(-3)^2$

$(-3)^2 = (-3)(-3) = 9.$

Note the difference between what you just did and the following:

Simplify -3^2

$-3^2 = -3 \cdot 3 = -9.$

In this last case, the square was only on the 3, not on the minus sign. The parentheses make all the difference in the world! Be careful with them. Continuing:

Simplify $(-3)^3$

$(-3)^3 = (-3)(-3)(-3)$
$= 9 \cdot (-3)$
$= -27.$

Simplify $(-3)^4$

$(-3)^4 = (-3)(-3)(-3)(-3)$
$= 9 \cdot 9$
$= 81.$

Note the pattern: A negative number taken to an even power gives a *positive* result (because the pairs of negatives cancel), and a negative number taken to an *odd* power gives a *negative* result (because, after canceling, there will be one minus sign left over). So if you are given you something slightly ridiculous like $(-1)^{1001}$, you know that the answer will either be +1 or –1, and, since 1001 is odd, the answer must be –1.

You can also do negatives with roots, but only if you're careful. You can do $\sqrt{16}$, because there is a number that squares to 16. That is,

$$\sqrt{16} = 4,$$

... because $4^2 = 16$. But what about $\sqrt{-16}$? Can you square any real number and have it come up *negative*? No! So you cannot take the square root (or the fourth root, or the sixth root, or the eighth root, or any other even root) of a negative number. On the other hand, you *can* do cube roots of negative numbers. For instance:

$$\sqrt[3]{-8} = -2,$$

because $(-2)^3 = -8$. For the same reason, you can take any odd root (third root, fifth root, seventh root, etc.) of a negative number.

Adapted from http://www.purplemath.com

Follow-up exercises

5. Write 10 questions to the text.

6. In groups of 3 or 4 ask and answer your questions. Refer to the text.

7. Write 4 mathematical operations. Don't show them to your partner. Dictate the operations to your partner. Check correctness when you finish.

8. Quiz each other in pairs. Say a word from the texts in English and your partner will tell you the Polish equivalent. Swap roles.

9. With your teacher revise the rules of question formation in English.
In pairs decide who of you is A, and who is B. Person A – work with the text in Appendix 1A, person B – text in Appendix 1B. Each of you should read the whole text carefully and explain the unknown words using a dictionary. You will then have a few minutes to prepare question about the gaps (missing information) in your texts. When you finish, work in pairs asking each other your questions to get the whole text. Then check the text with your teacher.

Now – try reading aloud:

Extension of the notion of power:

$a^0 = 1$ and $a^{-n} = 1/a^n$ for all $a \neq 0$.

Examples:

1. $a^3 : a^5 = a^{3-5} = a^{-2} = 1/a^2$.
2. $25 \cdot \left(\frac{1}{a}\right)^{-n} \cdot (2n)^0 \cdot 5^{-3} \cdot \left(\frac{a}{x}\right)^{-n} = 5^{2-3} \cdot a^n \cdot a^{-n} \cdot x^{-(-n)} = 5^{-1} \cdot a^0 \cdot x^n = \frac{x^n}{5}$.
3. $27a^4b^4 \cdot 56a^2b^{-3} \cdot 42a^{-2}b^3$
$= 3^3 \cdot a^4b^4 \cdot 7 \cdot 2^3 \cdot a^2b^{-3} \cdot 7 \cdot 2 \cdot 3 \cdot a^{-2}b^3 = 2^4 \cdot 3^4 \cdot 7^2 a^4b^4 = \left(2^2 \cdot 3^2 \cdot 7a^2b^2\right)^2$
4. What is the energy in kWh (1 kWh = $3.6 \cdot 10^{13}$ $gcm^2 s^{-2}$) corresponding to a mass defect of 2 mg?
$E = m \cdot c^2$ (E energy, m mass, c velocity of light, $3 \cdot 10^{10}$ cm/s).
One obtains $\frac{2 \cdot 10^{-3} \cdot (3 \cdot 10^{10})^2}{3.6 \cdot 10^{13}}$ kWh = $\frac{2 \cdot 9 \cdot 10^{-3} \cdot 10^{20}}{3.6 \cdot 10^{13}}$ kWh = $5 \cdot 10^4$ kWh.
Consequently, if 2 mg of a substance are transformed completely into energy, the energy liberated is 50 000 kWh.

Chapter 3

☺That math professor's marriage is falling apart!
"*No wonder! He's into scientific computing – and she is incalculable!*"☺

Linear algebra

Lead-in:

1. Discuss the definition of linear algebra with your partner.

2. Listen to the recording and repeat the words:
linear transformation, *linear equation*, *magnitude*, *scalar*, *higher order derivative*, *real numbers*, *complex numbers*, *n-tuple*, *addition*, *multiplication*, *determinant*, *eigenvector*, *differential calculus*, *invertible matrix*, *isomorphism.*

3. Read the text and find the above words in it.

Text:

Linear algebra is a branch of mathematics concerning the study of vectors, vector spaces (also called *linear spaces*), linear transformations, and systems of linear equations in finite dimensions. Vector spaces are a central theme in modern mathematics; thus, linear algebra is widely used in both abstract algebra and functional analysis. Linear algebra also has a concrete representation in analytic geometry and it is generalized in operator theory. It has extensive applications in the natural sciences and the social sciences, since nonlinear models can often be approximated by a linear model.

History

The history of modern linear algebra dates back to the years 1843 and 1844. In 1843, William Rowan Hamilton (from whom the term *vector* stems) discovered the quaternions. In 1844, Hermann Grassmann published his book *Die lineale Ausdehnungslehre*. Arthur Cayley introduced (2 × 2) matrices, one of the most fundamental linear algebraic ideas, in 1857.

Elementary introduction

Linear algebra had its beginnings in the study of vectors in Cartesian 2-dimensional and 3-dimensional space. A vector, here, is a directed line segment, characterized by both its magnitude represented by length, and its direction. Vectors can be used to represent physical entities such as forces, and they can be added and multiplied by scalars, thus forming the first example of a real vector space.

Modern linear algebra has been extended to consider spaces of arbitrary or infinite dimension. A vector space of dimension *n* is called an *n-space*. Most of the useful results from 2 and 3-space can be extended to these higher dimensional spaces. Although many people cannot easily visualize vectors in *n-space*, such vectors or *n*-tuples are useful in representing data. Since vectors, as *n*-tuples, are *ordered* lists of *n* components, it is possible to summarize and manipulate data efficiently in this framework. For example, in economics, one can create and use, say, 8-dimensional vectors or 8-tuples to represent the Gross National Product of 8 countries. One can decide to display the GNP of 8 countries for a particular year, where the countries' order is specified, for example, United States, United Kingdom, France, Germany, Spain, India, Japan, Australia by using a vector ($\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$, $\mathbf{v}_5$, $\mathbf{v}_6$, $\mathbf{v}_7$, $\mathbf{v}_8$) where each country's GNP is in its respective position.

A vector space (or linear space), as a purely abstract concept, which is the subject of many theorems, is part of abstract algebra, and well integrated into this discipline. Some striking examples of this are the group of invertible linear maps or matrices, and the ring of linear maps of a vector space. Linear algebra also plays an important part in analysis, notably, in the description of higher order derivatives in vector analysis and the study of tensor products and alternating maps.

A vector space is defined over a field, such as the field of real numbers or the field of complex numbers. Linear operators take elements from a linear space to another (or to itself), in a manner that is compatible with the addition and scalar multiplication given on the vector space(s). The set of all such transformations is itself a vector space. If a basis for a vector space is fixed, every linear transform can be represented by a table of numbers called a matrix. The detailed study of the properties and algorithms acting on matrices, including determinants and eigenvectors, is considered to be part of linear algebra.

One can say quite simply that the linear problems of mathematics – those that exhibit linearity in their behaviour – are those most likely to be solved. For example differential calculus does a great deal with linear approximation to functions. The difference from non-linear problems is very important in practice.

The general method of finding a linear way to look at a problem, expressing this in terms of linear algebra, and solving it, if need be by matrix calculations, is one of the most generally applicable in mathematics.

Some useful definitions and theorems

- Theorem: Every linear space has a basis. (This statement is logically equivalent to the axiom of choice.)
- Definition: A non-zero matrix *A* with *n* rows and *n* columns is invertible if there exists a matrix *B* that satisfies $AB = BA = I$, where *I* is the identity matrix.
- Theorem: A matrix is invertible if and only if its determinant is different from zero.
- Theorem: A matrix is invertible if and only if the linear transformation represented by the matrix is an isomorphism (see also invertible matrix for other equivalent statements).
- Definition: A matrix is positive semidefinite if and only if each of its eigenvalues is greater than or equal to zero.
- Definition: A matrix is positive definite if and only if each of its eigenvalues is greater than zero.

Adapted from Wikipedia

Follow-up exercises

4. Work out the definitions of the words from exercise 2 in pairs.

5. Read the definitions to the whole group and the group will guess the words.

6. What are the Polish equivalents of these terms?

7. Write 6 true/false sentences to the text. Read them in groups of four and decide if they are true or false. Correct the false ones.

8. Match these words with the definitions below:

A. linear algebra
B. vector
C. vector space
D. matrix

1. A table of numbers (representing linear transformations).
2. The branch of mathematics concerned with the study of vectors, vector spaces, linear transformations and systems of linear equations in finite dimensions.
3. A directed line segment characterized by both its magnitude represented by length, and its direction.
4. The set of all vector transformations.

9. Give the plural forms of the following words:

derivative	theory
matrix	length
determinant	entity
study	*n*-tuple
analysis	basis

10. Put the missing phrases in the gaps:

Phrases:

- in several variables,
- physical sciences
- important in all levels and all areas of
- (called the coefficient)
- the basic methods of manipulating
- may have infinite degree,
- equal to the degree of that variable
- a constant monomial,
- constructed from
- from one or more variables and constants,
- information about the operator's eigenvalues
- the simultaneous zero sets of
- the simplest algebraic

Polynomials

In mathematics, a polynomial is an expression that is constructed 1) .... using only the operations of addition, subtraction, multiplication, and constant positive whole number exponents. For example, $x^2 - 4x + 7$ is a polynomial.

Polynomials are 2) .... mathematics. They are used to solve word problems in basic algebra classes, which also cover 3) .... polynomials. They are used to approximate other functions in calculus and numerical analysis. Outside mathematics, the basic equations in economics and many 4) .... are polynomial equations.

Polynomials are built from terms called monomials, which consist of a constant 5) .... multiplied by one or more variables (these are usually represented by letters). Each variable may have a constant positive whole number exponent. The exponent on a variable in a monomial is 6) .... in that monomial. Since $x = x^1$, the degree of a variable without a written exponent is one. A monomial with no variables is called 7) .... or just a constant. The degree of a constant term is 0. The coefficient of a monomial may be any number, including fractions, 8) .... and negative numbers. A polynomial that is 9) .... one variable is called univariate.

One also speaks of polynomials 10) .... obtained by taking the ring of polynomials of a ring of polynomials: R[X,Y] = (R[X])[Y] = (R[Y])[X]. These are of fundamental importance in algebraic geometry which studies 11) .... several such multivariate polynomials.

Polynomials are frequently used to 12) .... about some other object. The characteristic polynomial of a matrix or linear operator contains 13) .... The minimal polynomial of an algebraic element records 14) .... relation satisfied by that element.

Other related objects studied in abstract algebra are formal power series, which are similar to polynomials but 15) .... and the rational functions, which are ratios of polynomials.

Now – try reading aloud:

1. *Associative law of addition*: $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.
2. *Commutative law of addition*: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.
3. *Existence of zero*: There exists an element $\mathbf{0}$ in V such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$.
4. *Existence of inverses*: To every $\mathbf{x}$ in V there exists an element $-\mathbf{x}$ in V such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$.
5. *Associative law of multiplication*: $a(b\mathbf{x}) = (ab)\mathbf{x}$.
6. *Unital law*: $1\mathbf{x} = \mathbf{x}$.
7. *First distributive law*: $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$.
8. *Second distributive law*: $(a + b)\,\mathbf{x} = a\mathbf{x} + b\mathbf{x}$.

The sum of vectors **x** and **y** plus vector **z** equals vector **x** plus the sum of vectors **y** and **z**
$(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.

Vector *x* multiplied by scalar *b* and then scalar *a* equals vector *x* multiplied by the product of scalars *a* and *b*
$a(b\mathbf{x}) = (ab)\mathbf{x}$.

The sum of vectors *x* and *y* multiplied by scalar *a* is equal to vector *x* multiplied by *a* plus vector *y* multiplied by *a*
$a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$.

☺"*Mathematicians never die – they only lose some of their functions!*"☺

Analytic Geometry

Lead-in:

1. What is analytic geometry?

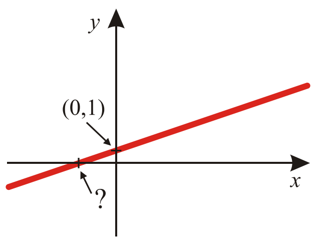
2. Listen to your teacher.

3. Listen and repeat:
graph, *equation*, *coordinate*, *axes*, *straight line*, *arrow*, *tangent*, *slope*, *steep line*, *variable*, *parabola*, *hyperbola*, *quadratic*, *intersection*, *curve*, *square root*, *angle*, *cosines.*

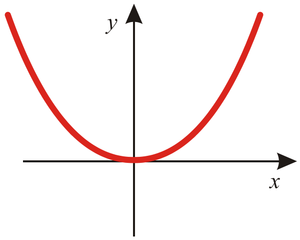
4. Read the text, find the words you don't know.

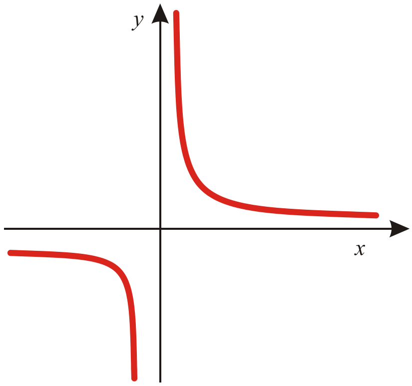
Text:

Analytic geometry is just a fancy name for graphing. You probably did plenty of it in algebra. It is a handy way to deal with equations.

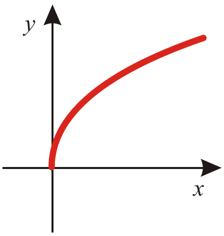


In this diagram, I have graphed a straight line $y = x/3 + 1$. You can draw a graph by trying a few values of x and y. For example, I tried $x = 0$ and I found that $y = 1$. There is a little arrow pointing to that point in the diagram. What do I get for x, when $y = 0$ (the question mark in the diagram)? Well, I get $x = –3$. I can keep on graphing every point of our equation, a time-consuming process. A computer program may do this for many values of x and y, and draw what looks like a continuous, straight line. But I observe that $y = x/3 + 1$ is a typical equation of a straight line. Then I only need to draw two points (0,1) and (–3,0), and then I can draw the line through these two points (Euclid assumed that two points determine a line). That line is the graph of our equation.
Our line has a slope of 1/3. The slope is the measure of how steep the line is. The graph goes up one for every three it goes to the right. On the freeway, we see signs that warn truck drivers of a 10% grade. This is a slope of 0.1 (which is 10%). That is not very steep for a line. But it is pretty steep for a freeway.





Here are the graphs corresponding to two slightly more complicated equations, a parabola $y = x^2$ and a hyperbola $y = 1/x$. We dealt with these in algebra. But, they caused us some problems, which you may not have noticed. It is difficult, using only algebra, to find the slope of these curves (at any given point). Why would we want to complicate our lives by finding the slope of a curve? Well, it is absolutely vital in physics and engineering. It is even of some interest to a skier.



This graph shows us something about functions (a function is a special kind of relation). This could be a graph of the equation $x = y^2$ ($y = x^2$ turned on its side). But such an equation is awkward for some uses. $y \neq \sqrt{x}$ is what we get, when we solve the equation $x = y^2$ for y. For most values of x, we get two values of y. This is perfectly acceptable in analytic geometry.

Adapted from http://www.jimloy.com and http://www.answers.com

Follow-up exercises

5. Match the words with their definitions:

A. graph,	E. slope,
B. hyperbola,	F. straight line,
C. equation,	G. curve,
D. steep line,	H. square root.

1. A picture which shows how two sets of information or variable amounts are related, usually by lines or curves.
2. Mathematical statement in which you show that two amounts are equal using mathematical symbols.
3. A surface which lies at an angle to the horizontal so that some points on it are higher than others.
4. A line which bends continuously and has no straight parts.
5. The number that was multiplied by itself to reach that number: “The …. of 49 is 7”.
6. A line continuing in one direction without bending or curving.
7. A line rising or falling at a sharp angle.
8. A curve whose ends continue to move apart from each other.

6. Are these statements true or false according to the text?
1. To determine a line we need at least three points.
2. The slope is the measure of how steep a line is.
3. It is easy, using only algebra, to find the slope of a parabola or a hyperbola.
4. Finding a slope of a curve is important in physics and engineering.

7. Read the text and fill in the gaps with the following words:
geometry, equations, planes, dimensions, numerical, vector, analytic, coordinate, circles, shapes.

Analytic geometry, also called 1) ………. **geometry** and earlier referred to as **Cartesian geometry**, is the study of 2) ………. Using the principles of algebra. Usually the Cartesian coordinate system is applied to manipulate 3) ……….. for 4) ………… lines, curves, and 5) ……….. often in two and sometimes in three 6) ……….. of measurement. As taught in school books, analytic geometry can be explained more simply: it is concerned with defining geometrical 7) ……….. in a numerical way, and extracting 8) ……….. information from that representation. The numerical output, however, might also be a 9) ……….. or a shape. Some consider that the introduction of 10) ……….. geometry was the beginning of modern mathematics.

8. Read the text below. Write a summary with your partner, inserting 10 factual mistakes.

Analytic geometry: branch of **geometry** in which points are represented with respect to a coordinate system, such as Cartesian coordinates, and in which the approach to geometric problems is primarily algebraic. Its most common application is in the representation of equations involving two or three variables as curves in two or three dimensions or surfaces in three dimensions. For example, the linear equation $ax + by + c = 0$ represents a straight line in the xy-plane, and the linear equation $ax + by + cz + d = 0$ represents a plane in space, where a, b, c, and d are constant numbers (coefficients). In this way a geometric problem can be translated into an algebraic problem and the methods of algebra brought to bear on its solution. Conversely, the solution of a problem in algebra, such as finding the roots of an equation or system of equations, can be estimated or sometimes given exactly by geometric means, e.g., plotting curves and surfaces and determining points of intersection.

In plane **analytic geometry** a line is frequently described in terms of its slope, which expresses its inclination to the coordinate axes; technically, the slope m of a straight line is the (trigonometric) tangent of the angle it makes with the x-axis. If the line is parallel to the x-axis, its slope is zero. Two or more lines with equal slopes are parallel to one another. In general, the slope of the line through the points (x_1, y_1) and (x_2, y_2) is given by $m = (y_2 - y_1) / (x_2 - x_1)$. The conic sections are treated in **analytic geometry** as the curves corresponding to the general quadratic equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$, where $a, b, \dots, f$ are constants and a, b, and c are not all zero.

In solid **analytic geometry** the orientation of a straight line is given not by one slope but by its direction cosines, λ, μ, and ν, the cosines of the angles the line makes with the x-, y-, and z-axes, respectively; these satisfy the relationship $\lambda^2 + \mu^2 + \nu^2 = 1$. In the same way that the conic sections are studied in two dimensions, the 17 quadric surfaces, e.g., the ellipsoid, paraboloid, and elliptic paraboloid, are studied in solid **analytic geometry** in terms of the general equation $ax^2 + by^2 + cz^2 + dxy + exz + fyz + px + qy + rz + s = 0$.

The methods of **analytic geometry** have been generalized to four or more dimensions and have been combined with other branches of geometry**. Analytic geometry** was introduced by René Descartes in 1637 and was of fundamental importance in the development of the calculus by Sir Isaac Newton and G.W. Leibniz in the late 17th century. More recently it has served as the basis for the modern development and exploitation of algebraic geometry.

9. Make groups of four. One pair reads the summary. The other pair should listen and write down the mistakes. When you finish compare with the text. Swap roles.

Now – try reading aloud:

Inner products of the basis vector:

$$\mathbf{i}\cdot\mathbf{i}=\mathbf{j}\cdot\mathbf{j}=\mathbf{k}\cdot\mathbf{k}=1$$

$$\mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{i}=0\,.$$

Inner product **a·b** in coordinates:
If

$$\mathbf{a}=a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k}\ ,$$

and

$$\mathbf{b}=b_x\mathbf{i}+b_y\mathbf{j}+b_z\mathbf{k}\ ,$$

then

$$\mathbf{a}\cdot\mathbf{b}=a_xb_x+a_yb_y+a_zb_z\ .$$

Norm of **a**:

$$|\mathbf{a}|=\sqrt{a_1^2+a_2^2+\cdots+a_n^2}=\sqrt{a\cdot a}$$

Generalized triangle inequality:

$$|\mathbf{x}_1+\mathbf{x}_2+\cdots+\mathbf{x}_n|\le|\mathbf{x}_1|+|\mathbf{x}_2|+\cdots+|\mathbf{x}_n|\ .$$

Cauchy-Schwarz inequality:

$$\left(\sum_{i=1}^{n}a_ib_i\right)^2\le\left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right)$$

Angles, if **a** and **b** are two non zero vectors, then:

$$\cos\angle(\mathbf{a},\mathbf{b})=\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}=\frac{a_xb_x+a_yb_y+a_zb_z}{\sqrt{\left(a_x^2+a_y^2+a_z^2\right)\left(b_x^2+b_y^2+b_z^2\right)}}$$

Inner product rules in R^n:
For any vectors **a**, **b**, **c** $\in R^n$ we have

$$\mathbf{a}\cdot\mathbf{b}=(a_1,a_2,\ldots,a_n)\cdot(b_1,b_2,\ldots,b_n)=a_1b_1+a_2b_2+\cdots+a_nb_n=\mathbf{b}\cdot\mathbf{a}\ ,$$

$$\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$$

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}\ ,$$

$$d(\mathbf{a}\cdot\mathbf{b})=(d\mathbf{a})\cdot\mathbf{b}=\mathbf{a}\cdot(d\mathbf{b}),\ \ d\in R\ .$$

1. *y* equals *x* divided by 3 plus 1

   $y = x/3 + 1.$

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2. *y* equals *x* squared

   $y = x^2.$

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3. *y* equals the inverse of *x*

   $y = 1/x.$

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4. *m* equals *y* 2 minus *y* 1 divided by *x* 2 minus *x* 1

   $m = (y_2 - y_1) / (x_2 - x_1).$

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5. *a* times *x* squared plus *b* times *x* times *y* plus *c* times *y* squared plus *d* times *x* plus *e* times *y* plus *f* equals 0

   $ax^2 + bxy + cy^2 + dx + ey + f = 0.$

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6. lambda squared plus mi squared plus ni squared equals 1

   $\lambda^2 + \mu^2 + \nu^2 = 1.$

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7. *a* times *x* squared plus *b* times *y* squared plus *c* times *z* squared plus *d* times *x* times *y* plus *e* times *x* times *z* plus *f* times *y* times *z* plus *p* times *x* plus *q* times *y* plus *r* times z plus *s* equals 0

   $ax^2 + by^2 + cz^2 + dxy + exz + fyz + px + qy + rz + s = 0.$

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8. The scalar product of vector **i** times itself equals the scalar product of vector **j** times itself equals … equals 1

   $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1.$

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9. Vector **a** equals scalar *a* sub *x* times **i** plus scalar *a y* times **j** plus scalar *a z* times **k**

   $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}.$

---

10. Vector **b** equals scalar *b* sub *x* times **i** plus scalar *b y* times **j** plus scalar *b z* times **k**

    $\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}.$

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11. The cosine of an angle between vectors **a** and **b** is equal to their scalar product divided by a product of their lengths

$$\cos\angle(\mathbf{a},\mathbf{b}) = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{\left(a_x^2 + a_y^2 + a_z^2\right)\left(b_x^2 + b_y^2 + b_z^2\right)}}\,.$$

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12. The scalar product of vectors **a** and **b**, where *a* and *b* have coordinates *a* 1 to *a n* and *b* 1 to *b n* respectively is the sum of products *a* 1 times *b* 1 to *a n* times *b n* and is equal to the scalar product of vectors **b** and **a**

$$\mathbf{a}\cdot\mathbf{b}=(a_1,a_2,\ldots,a_n)\cdot(b_1,b_2,\ldots,b_n)=a_1b_1+a_2b_2+\cdots+a_nb_n=\mathbf{b}\cdot\mathbf{a}.$$

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13. The scalar product of vectors **a** and the sum of **b** and **c** is the scalar product of **a** and **b** plus the scalar product of **a** and **c**

$$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}.$$

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14. The scalar product of vectors **a** and **b** multiplied by scalar *d* is equal to the scalar product of **a** times ***d*** and ***b*** and is also equal to the scalar product of **a** and **b** times ***d***

$$\boldsymbol{d}(\mathbf{a}\cdot\mathbf{b})=(\boldsymbol{d}\mathbf{a})\cdot\mathbf{b}=\mathbf{a}\cdot(\boldsymbol{d}\mathbf{b}).$$

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15. The length of a is the square root of the sum of its squared coordinates

$$|\mathbf{a}|=\sqrt{a_1^2+a_2^2+\cdots+a_n^2}.$$

---

16. The length of the sum of vectors **x** 1 to **x** *n* is less or equal than the sum of the lengths of these vectors

$$|\mathbf{x}_1+\mathbf{x}_2+\cdots+\mathbf{x}_n|\le|\mathbf{x}_1|+|\mathbf{x}_2|+\cdots+|\mathbf{x}_n|.$$

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17. The absolute value of the scalar product of **a** and **b** is less or equal than the product of the lengths of these vectors

$$|\mathbf{a}\cdot\mathbf{b}|\le|\mathbf{a}||\mathbf{b}|=\sqrt{\sum_{i=1}^{n}a_i^2}\sqrt{\sum_{i=1}^{n}b_i^2}.$$

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18. The cosine of the angle between vectors **a** and **b** is equal to their scalar product divided by the product of their lengths and the angle is between 0 and pi

$$\cos\angle(\mathbf{a},\mathbf{b})=\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|},\qquad 0\le\angle(\mathbf{a},\mathbf{b})\le\pi.$$

Chapter 5

☺*The mother of already three is pregnant with her fourth child.*
One evening, the eldest daughter says to her dad:
"Do you know, daddy, what I've found out?"
"No." "The new baby will be Chinese!" "What?!"
"Yes. I've read in the paper that statistics shows that every fourth child born nowadays is Chinese..."☺

Mathematical Analysis

Lead-in:

1. Warm up: What is a function?

2. Listen and repeat:

domain, *co-domain*, *argument*, *correspondence*, *monotone*, *increasing*, *bounded*, *discontinuous.*

3. Read the text and find out more about the basic notions of functions.

Text:

Domain and codomain of a function. In elementary mathematics we study functions only in a set of real numbers ***R***. This means that an argument of a function can take on only those real values, at which a function is defined, i.e. it also takes on only real values. A set ***X*** of all admissible real values of an argument *x*, at which a function $y = f(x)$ is defined, is called a *domain of the function f*. A set ***Y*** of all real values *y*, that the function adopts, is called a *codomain of a function*. Now we can formulate a definition of a function more exactly: *such a rule* (*law*) *of a correspondence between elements of a set* ***X*** *and a set* ***Y***, *that for each element of a set* ***X*** *one and only one element of a set* ***Y*** *can be found, is called a function*. From this definition it follows, that a function is given if:

- the domain ***X*** of a function is given;
- the codomain ***Y*** of a function is given;
- the correspondence rule (law), is known.

A correspondence rule must be such, that for *each value of an argument only one value of a function* can be found. This requirement of a single-valued function is obligatory.

Examples:

$y = x^3 + 1$ the function domain: $x \in \boldsymbol{R}$,

the function codomain: $y \in \boldsymbol{R}$,

$y = \sqrt{x-5}$ the function domain: $x \geq 5$

the function codomain: $y \geq 0$,

$y = \dfrac{\sin^2 x}{|x-4|}$ the function domain: $x \neq 4$

the function codomain: $y \geq 0$.

Monotone function. If for any two values of an argument x_1 and x_2 from the condition $x_2 > x_1$ it follows $f(x_2) > f(x_1)$, then a function is called *increasing*; if for any x_1 and x_2 from the condition $x_2 > x_1$ it follows $f(x_2) < f(x_1)$, then a function is called *decreasing*. A function, which only increases or only decreases, is called a *monotone function*.

Bounded and unbounded functions. A function is *bounded*, if there exists a positive number M such that $|f(x)| \leq M$ for all values of x. If such positive number does not exist, then this function is *unbounded*.

***Examples*:**

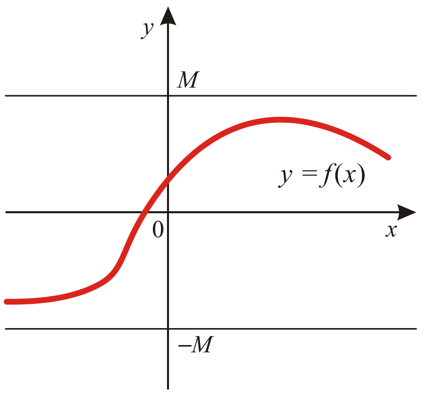


Fig. 5.1

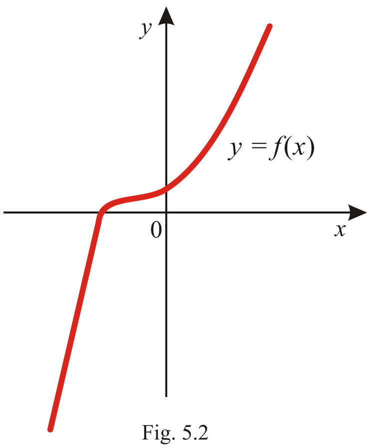


Fig. 5.2

A function, shown in Fig. 5.1, is a bounded, but not a monotone function. In Fig. 5.2 quite the opposite, we see a monotone, but unbounded function.

Continuous and discontinuous functions. A function $y = f(x)$ is called a continuous function at a point $x = a$, which belongs to the function $f(x)$ domain, if:

1) the function is defined at $x = a$, i.e. $f(a)$ exists;

2) a *finite* $\lim\limits_{x \to a} f(x)$ exists;

3) $f(a) = \lim\limits_{x \to a} f(x)$.

If one of the last two conditions is not satisfied, this function is called *discontinuous* at the point $x = a$.

If a function is continuous at *all* points of its domain, it is called a *continuous function*.

***Examples*:**

1. The function $y = \begin{cases} \dfrac{3x}{x-5}, & x \neq 5 \\ 0, & x = 5 \end{cases}$ is discontinuous at $x = 5$

   At all the rest of the points it is a continuous function.

2. The function $y = \begin{cases} 1, x \neq 0 \\ 0, x = 0 \end{cases}$ is discontinuous at $x = 0$.

Even and odd functions. If for *every* x from a function domain: $f(-x) = f(x)$, then this function is called *even*; if $f(-x) = -f(x)$, then this function is called *odd.* A graph of an even function is symmetrical relatively to y-axis (Fig. 5.3), a graph of an odd function is symmetrical relatively to the origin of coordinates (Fig. 5.4).

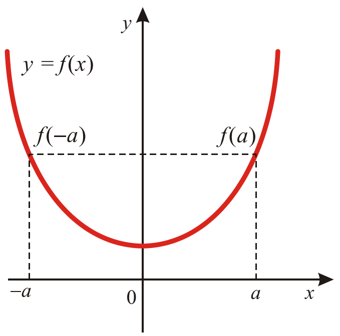


Fig. 5.3

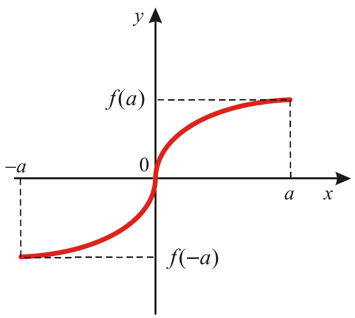


Fig. 5.4

Periodic function. A function $f: X \to Y$ is *periodic*, if there is a positive number $T > 0$, such that for every value of x holds $x \pm T \in X$ and $f(x + T) = f(x)$. The *least* such number T is called *period of function*. All trigonometric functions are periodic.

***Example* 1**. Prove that $y = \sin x$ has number 2π as a period.

Solution:

Let us notice that the domain of function $y = \sin(x)$ is the set of real numbers $\boldsymbol{R}$.
We know, that $\sin(x + 2\pi n) = \sin x$, where $n = 0, \pm 1, \pm 2,$
Hence, adding $2\pi n$ to an argument of a sine doesn't change its value.
Maybe another number with such property exists?
Assume that P is such number, i.e. the equality:

$$\sin(x + P) = \sin x,$$

is valid for any value of x. Then this is valid for $x = \pi/2$, i.e.

$$\sin(\pi/2 + P) = \sin \pi/2 = 1.$$

But $\sin(\pi/2 + P) = \cos P$ according to the reduction formula. Then from the two last expressions it follows, that $\cos P = 1$, but we know, that this equality is right only if $P = 2\pi n$. Because the least non-zero number of $2\pi n$ is 2π, this is a period of $y = \sin x$. It is proved analogously, that 2π is also a period for $y = \cos x$.

Prove that functions $y = \tan x$ and $y = \cot x$ have π as a period.

***Example* 2**. Which number is a period for the function $y = \sin 2x$?

Solution:

Consider

$$\sin 2x = \sin (2x + 2\pi n) = \sin [\, 2\, (x + \pi n)].$$

We see, that adding πn to an argument x, doesn't change the function value. The least non-zero number of πn is π, so this is a period of $\sin 2x$.

Zeros of function. An argument value, at which a function is equal to zero, is called a *zero* (*root*) *of the function*. It is possible that a function has a greater number of zeros. For instance, the function $y = x\,(x + 1)\,(x - 3)$ has three zeros: $x = 0$, $x = -1$, $x = 3$. Geometrically, a zero of a function is x-coordinate of a point of intersection of the function graph and x-axis. In Fig. 5.5 a graph of a function with zeros $x = a$, $x = b$ and $x = c$ is represented.

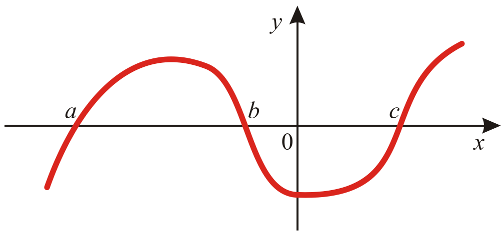


Fig. 5.5

An **asymptote** of a real-valued function $y = f(x)$ is a curve which describes the behaviour of f as either x or y tends to ininity. If an asymptote is parallel with the y-axis, we call it a vertical asymptote. If an asymptote is parallel with the x-axis, we call it a horizontal asymptote. When an asymptote is not parallel with the x- or y-axis, it is called an oblique asymptote (another name for an oblique asymptote is a slant asymptote). A function can have more than one asymptote.

Asymptotes are formally defined using limits.

Follow-up exercises

4. Write 6 true/false sentences to the text.

5. Work in pairs. Read your sentences to your partner and ask them to decide if they are true or false. Refer to the text if necessary. Swap roles.

6. Work in groups of 4. One of you is A, the others are B, C, D.
Person A – read text A below,
Person B – read text B,
Person C – read text C,
Person D – read text D.

Prepare short summaries of your texts.

7. Work in your groups of 4 again – read your summaries to the group, explain the most important terms together.

8. Make a list of 10 important terms from this chapter. Quiz other students in groups of three.

9. Work in pairs. One of you should draw a graph, and the other person should try and name it. Swap roles.

10. Close the book. Write definitions of the following notions:
- domain of function,
- linear function,
- hyperbola,
- quadratic function,
- power function.

Texts:

Elementary functions and their graphs:

Text A:

1. ***Proportional values***. If variables y and x are *directly proportional*, then the functional dependence between them is represented by the equation:

$$y = kx,$$

where k is a constant a factor of proportionality.

A graph of a direct proportionality is a straight line, going through an origin of coordinates and forming with an x-axis an angle α, a tangent of which is equal to k: $\tan \alpha = k$ (Fig. 5.6). Therefore, a factor of proportionality is called also a *slope*. There are shown three graphs with $k = 1/3$, $k = 1$ and $k = -3$ in Fig. 5.6.

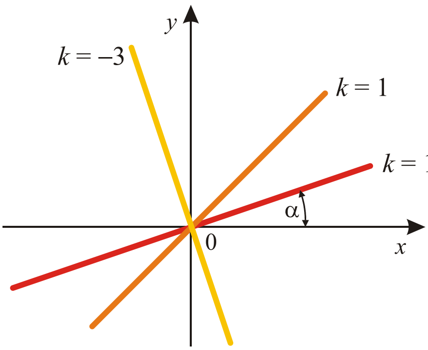


Fig. 5.6

2. ***Linear equation*** (***general form***). If variables y and x are tied by the 1st degree equation:

$$A x + B y = C,$$

(at least one of the numbers A or B is non-zero), then a graph of the equation is a straight line. If $C = 0$, then it goes through an origin of coordinates, otherwise – it does not. Graphs of linear equations for different combinations of *A*, *B*, *C* are represented in Fig. 5.7.

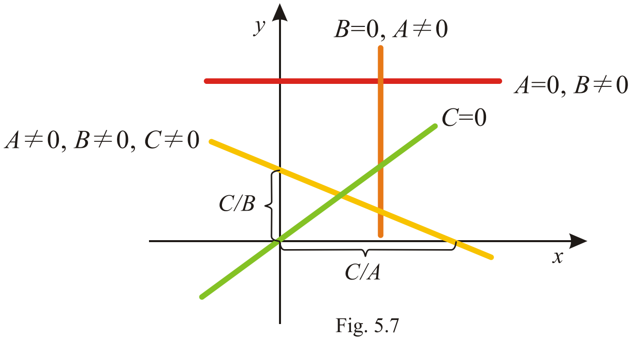


Fig. 5.7

3. ***Inverse proportionality***. If variables *y* and *x* are *inversely proportional*, then the functional dependence between them is represented by the equation:

$$y = k / x ,$$

where k is a constant.

A graph of an inverse proportionality is a curve, having two branches (Fig. 5.8). Such a graph is called a *hyperbola*. These curves are obtained by crossing a circular cone by a plane. As shown in Fig. 5.8, a product of coordinates of a hyperbola points is a constant value, equal in this case to 1. In general this value is *k*, as it follows from a hyperbola equation:

$$x\,y = k.$$

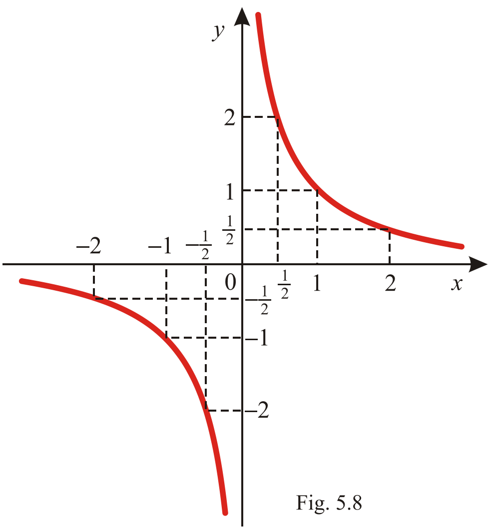


Fig. 5.8

The main characteristics and properties of hyperbola:

— the function domain: $x \neq 0$, and codomain: $y \neq 0$;

— the function is monotone (decreasing) at $x < 0$ and at $x > 0$, but it is not monotone on the whole domain;

— the function is unbounded, odd, non-periodic;
— there are no zeros of the function.

Text B:

4. ***Quadratic function***. This is the function: $y = ax^2 + bx + c$, where a, b, c are constants, $a \neq 0$. In the simplest case we have $b = c = 0$ and $y = ax^2$. A graph of this function is a *quadratic parabola* – a curve, going through an origin of coordinates (Fig. 5.9). Every parabola $y = ax^2$ has an axis of symmetry *OY*, which is called an *axis of parabola*. The point *O* of intersection of a parabola $y = ax^2$ with its axis is a *vertex of parabola*.

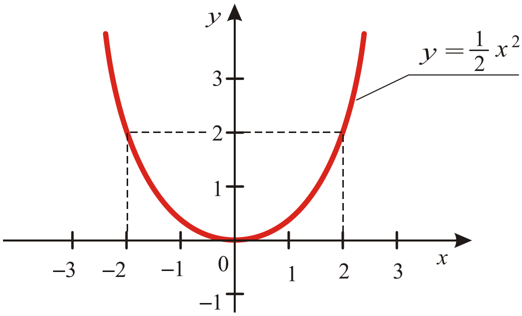


Fig. 5.9

A graph of the function $y = ax^2 + bx + c$ is also a quadratic parabola of the same shape, that $y = ax^2$, but its vertex is not an origin of coordinates, this is a point with coordinates:

$$\left(-\frac{b}{2a},\quad -\frac{\Delta}{4a}\right)$$

The form and location of a quadratic parabola in a coordinate system depends completely on two parameters: the coefficient a of x^2 and *discriminant* Δ: ($\Delta = b^2 - 4ac$). These properties follow from analysis of the quadratic equation roots. Some possible different cases for a quadratic parabola are shown in Fig. 5.10.

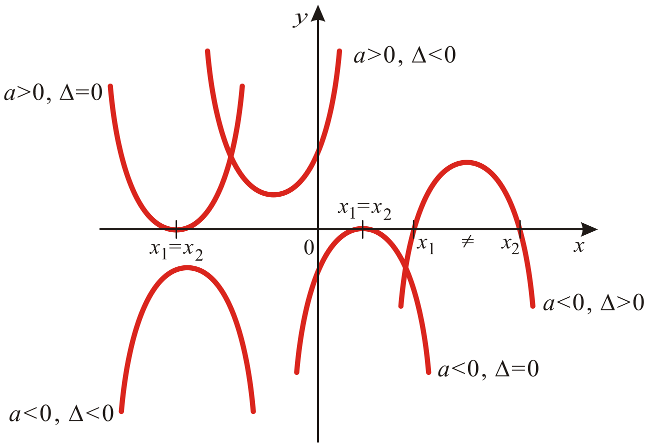


Fig. 5.10

The main characteristics and properties of a quadratic parabola:
— the function domain: $-\infty < x < +\infty$ (i.e. x is any real number)
— the function is not monotone on the whole, but to the right or to the left of the vertex it behaves as a monotone function;
— the function is unbounded, continuous everywhere, and non-periodic;
— the function has no zeros at $\Delta < 0$. (What about $\Delta \geq 0$?).

5. ***Power function.*** This is the function: $y = ax^n$ where a, n are constants. For $n = 1$ we obtain the function, called a *direct proportionality*: $y = ax$; for $n = 2$ – a *quadratic parabola*; for $n = -1$ – an *inverse proportionality* or a *hyperbola*. So, these functions are particular cases of a power function. We know, that a zero power of every non-zero number is 1, thus for $n = 0$ the power function becomes a constant: $y = a$, i.e. its graph is a straight line, parallel to an x-axis. All these cases (for $a = 1$) are shown in Fig. 5.11 ($n \geq 0$) and Fig. 5.12 ($n < 0$). Negative values of x are not considered here, because then some of the functions:
eg. $y = x^{1/2} = \sqrt{x}$, $y = x^{1/4} = \sqrt[4]{x}$ lose their meaning.

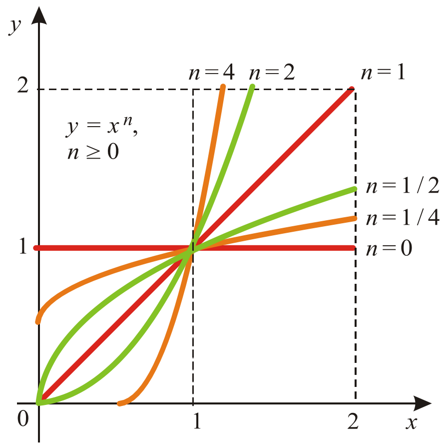


Fig. 5.11

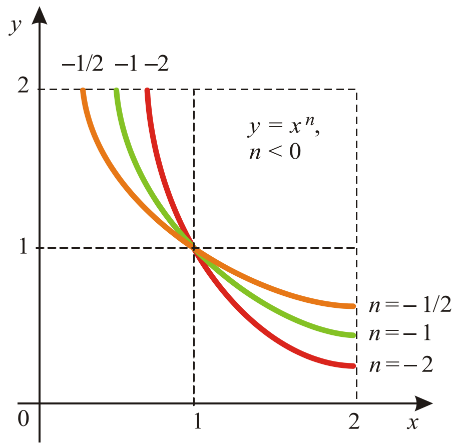


Fig. 5.12

If n is an integer, power functions also have a meaning for $x < 0$, but their graphs have different forms depending on whether n is an even or an odd number. Two of such power functions are shown in Fig. 5.13: for $n = 2$ and $n = 3$.
For $n = 2$ the function is even and its graph is symmetric relatively to an axis Y; for $n = 3$ the function is odd and its graph is symmetric relatively to an origin of coordinates. The function $y = x^3$ is called a *cubic parabola*.

In Fig. 5.14 the (two-valued) function $y = \pm\sqrt{x}$ is represented. This function is inverse to the quadratic parabola $y = x^2$, its graph is obtained by rotating the quadratic parabola graph around a bisector of the 1-st coordinate angle. (This is the way to obtain a graph of every inverse function from its original function). We see from the graph, that this is the two-valued function (the sign ± before the square root symbol says that). Such functions are not studied in elementary mathematics, therefore we usually consider one of its branches as a function: either an upper or a lower branch.

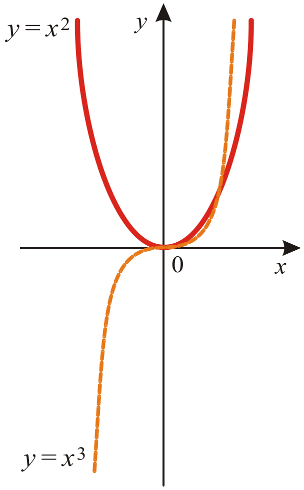


Fig. 5.13

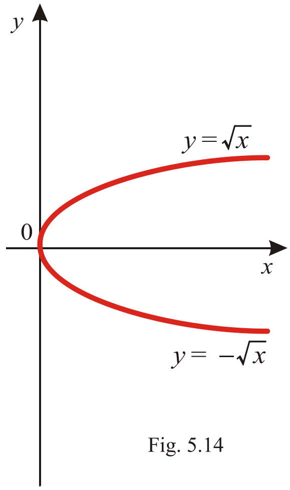


Fig. 5.14

Text C:

6. ***Exponential function***. The function $y = a^x$, where a is a positive constant number, is called an *exponential function.* The argument x takes on *any real* values; *only positive numbers* are considered to be the function values. Graphs of an exponential function for $a = 2$ and $a = 1/2$ are shown in Fig. 5.15. They are all going through the point (0, 1). For $a = 1$ we have a straight line as a graph, parallel to x-axis, i.e. the function is a constant value, equal to 1. For $a > 1$ an exponential function increases, and for $0 < a < 1$ it decreases.

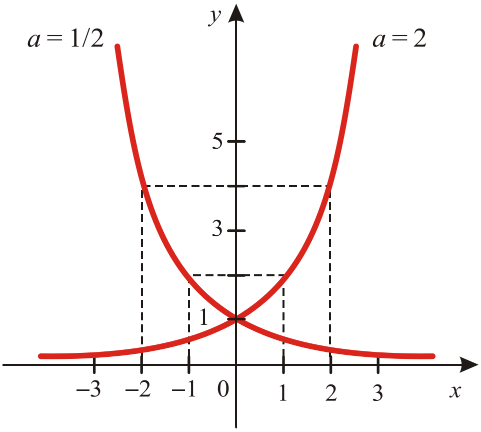


Fig. 5.15

The main characteristics and properties of an exponential function:

— the function domain: $-\infty < x < +\infty$ (i.e. x is any real number) and its codomain: $y > 0$;

— this is a monotone function: it increases if $a > 1$ and decreases if $0 < a < 1$;
— the function is unbounded, continuous everywhere, non-periodic;
— the function has no zeros.

7. ***Logarithmic function***. The function $y = \log_a x$, where a is a positive constant number not equal to 1, is called a *logarithmic function*. This is an inverse function relatively to an exponential function; its graph (Fig. 5.16) can be obtained by rotating a graph of an exponential function around the bisector of the 1-st coordinate angle.

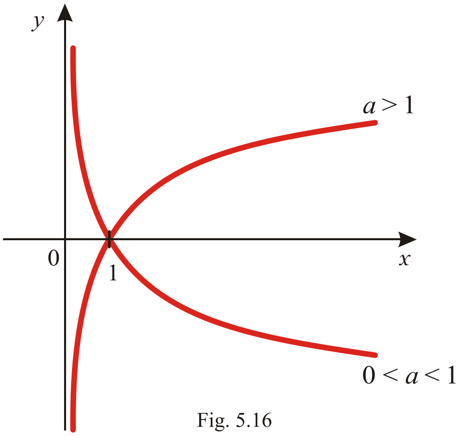


Fig. 5.16

The main characteristics and properties of a logarithmic function:
— the function domain: $x > 0$ and its codomain: $-\infty < y < +\infty$ (i.e. y is any real number);
— this is a monotone function: it increases for $a > 1$ and decreases for $0 < a < 1$;
— the function is unbounded, continuous everywhere, non-periodic;
— the function has one zero: $x = 1$.

Text D:

8. ***Trigonometric functions.*** Building trigonometric functions, we use a *radian* as a measure of angles. Then the function $y = \sin x$ is represented by the graph shown in (Fig. 5.17). This curve is also called a *sinusoid*.

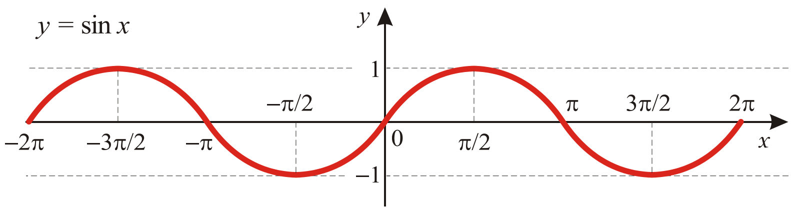


Fig. 5.17

The graph of the function $y = \cos x$ is represented in Fig. 5.18 ; this is also a sinusoid, obtained from the graph of $y = \sin x$ by shifting it along the x-axis to the left by $\pi / 2$.

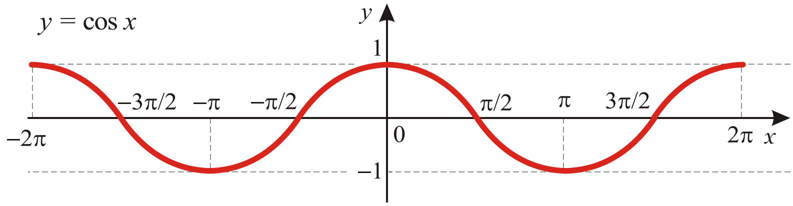


Fig. 5.18

From these graphs the following main characteristics and properties of the functions are straightforward:

— the functions have a domain: $-\infty < x < +\infty$ and a codomain: $-1 \le y \le 1$

— these are periodic functions: their period is 2π;

— the functions are bounded ($|y| \le 1$), continuous everywhere; they are not monotone functions, but there are so called *intervals of monotony*, inside of which they behave as monotone functions (see graphs in Fig. 19 and Fig. 20);

— the functions have an innumerable set of zeros.

Graphs of functions $y = \tan x$ and $y = \cot x$ are shown in Fig. 5.19 and Fig. 5.20 correspondingly.

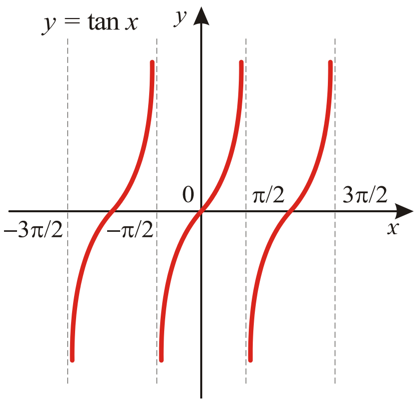


Fig. 5.19

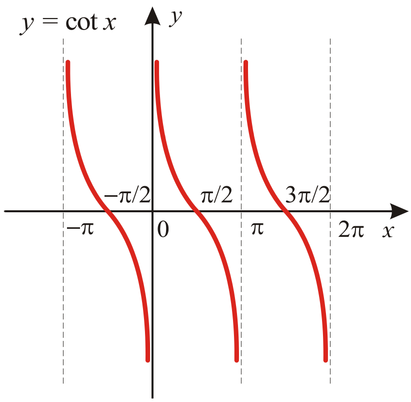


Fig. 5.20

The graphs show, that these functions are: periodic (their period is π), unbounded, and they have the intervals of monotony. The domain and codomain of these functions are:

for $y = \tan x$: $x \ne \pi/2 + k\pi$, $k = 0$, ± 1, ± 2, …; $-\infty < y < +\infty$

for $y = \cot x$: $x \ne k\pi$, $k = 0$, ± 1, ± 2, …; $-\infty < y < +\infty$

9. ***Inverse trigonometric functions***. We will give only short comments concerning their graphs obtained by rotating the graphs of trigonometric functions around a bisector of the 1-st coordinate angle.

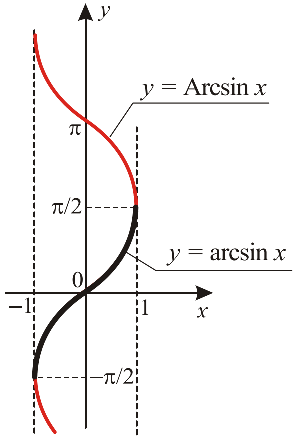


Fig. 5.21

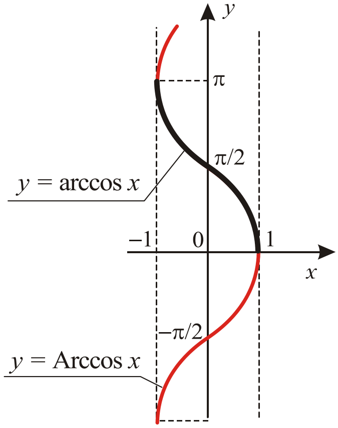


Fig. 5.22

The functions $y = \mathrm{Arc}\sin x$ (Fig. 5.21) and $y = \mathrm{Arc}\cos x$ (Fig. 5.22) are multivalued, unbounded functions; their domain and codomain are correspondingly: $-1 \le x \le 1$ and $-\infty < y < +\infty$. Because they are multi-valued functions, not considered in an elementary mathematics, their principal values $y = \mathrm{arc}\sin x$ and $y = \mathrm{arc}\cos x$ are considered as inverse trigonometric functions; their graphs have been distinguished in Fig. 5.21 and Fig. 5.22 as bold lines.

The functions $y = \mathrm{arc}\sin x$ and $y = \mathrm{arc}\cos x$ have the following characteristics and properties:

— both functions have the same domain: $-1 \le x \le 1$; their codomains are: $-\pi/2 \le y \le \pi/2$ for $y = \mathrm{arc}\sin x$ and $0 \le y \le \pi$ for $y = \mathrm{arc}\cos x$;
— they are bounded, non-periodic, continuous and monotone functions ($y = \mathrm{arc}\sin x$ is an increasing function; while $y = \mathrm{arc}\cos x$ is a decreasing function);
— each of the functions has one zero ($x = 0$ for $y = \mathrm{arc}\sin x$; $x = 1$ for $y = \mathrm{arc}\cos x$).

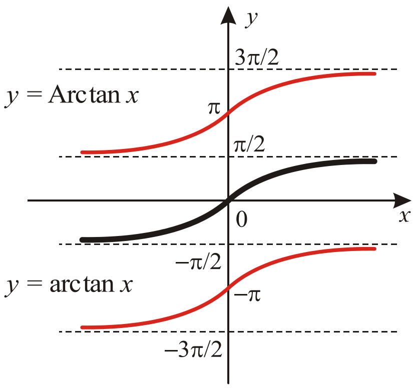


Fig. 5.23

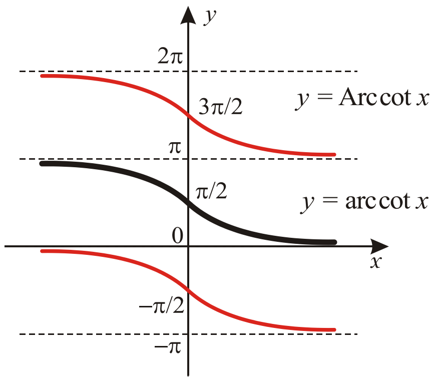


Fig. 5.24

The functions $y = \mathrm{Arc}\tan x$ (Fig. 5.23) and $y = \mathrm{Arc}\cot x$ (Fig. 5.24) are multivalued, unbounded functions; their domains are the same: $-\infty < x < +\infty$. Their principal values $y = \mathrm{arc}\tan x$ and $y = \mathrm{arc}\cot x$ are considered as inverse trigonometric functions; their graphs have been distinguished in Fig. 25 and Fig. 26 as bold branches.

The functions $y = \operatorname{arc}\tan x$ and $y = \operatorname{arc}\cot x$ have the following characteristics and properties:

— both functions have the same domain: $-\infty < x < +\infty$; their codomains are: $-\pi/2 < y < \pi/2$ for $y = \operatorname{arc}\operatorname{tg} x$, and $0 < y < \pi$ for $y = \operatorname{arc}\cot x$;

— they are bounded, non-periodic, continuous and monotone functions ($y = \operatorname{arc}\tan x$ is an increasing function, $y = \operatorname{arc}\cot x$ is a decreasing function);

— only $y = \operatorname{arc}\tan x$ has one zero ($x = 0$); $y = \operatorname{arc}\cot x$ has no zeros.

Adapted from www.bymath.com

Now – try reading aloud:

1. $x = e\dfrac{\sin\tau}{\sin\varepsilon} = 2.549$ miles.

   x equals e times sine of tau over sine of epsilon equals two point five four nine miles.

---

2. $\dfrac{\mathrm{d}x}{\mathrm{d}y} = f$

   d *x* by d *y* equals *f*.

---

3. $x = \xi + \int\limits_{\eta}^{y} y\mathrm{d}y = \xi + \frac{1}{2}(y^2 - \eta^2)$ for $y > 0$.

   x equals xi plus integral from eta to *y* of *y* d *y* equals xi plus 1 over two times *y* squared minus eta squared

---

4. $y = \sqrt{\eta^2 + 2(x - \xi)}$ for $x > \xi - \eta^2/2$.

   y equals square root of eta squared plus two times *x* minus xi; for *x* greater than xi minus eta squared over two.

---

5. $e^{i\theta} = \cos\theta + i\sin\theta$

   e to the power of *i* times theta equals cosine of theta plus *i* times sine of theta.

---

6. $e^{i\pi} + 1 = 0$

   e to the power of *i* times pi plus one equals zero.

---

7. $\varsigma(2) = \sum\limits_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6}$

   function zeta of two equals the sum from *n* equal one to infinity of one divided by *n* squared.

---

8. $\gamma = \lim\limits_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} - \ln(n)\right)$.

   gamma is a limit of a sum 1 plus 1 over 2 plus 1 over 3 plus one over 4 plus and so on plus 1 over *n* minus natural logarithm of *n* as *n* goes to plus infinity.

☺Q: What does the zero say to the eight?
A: Nice belt!☺

Geometry. Part 1

Lead-in:

1. Warm up: What geometrical terms do you know?

2. Repeat after the recording:
line, *geometry*, *intersection*, *ray*, *endpoint*, *parallel*, *perpendicular*, *angle*, *vertex*, *degree*, *acute angle*, *obtuse angle*, *angle bisector*.

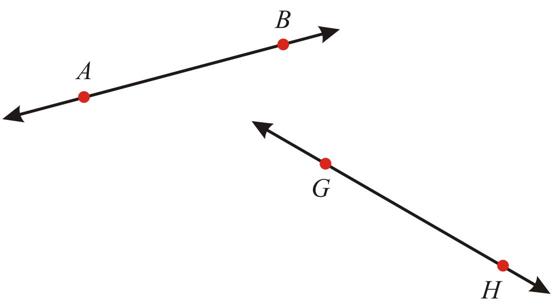
3. Read the text. Write down any other terms connected with geometry.

Text:

Lines

A line is one of the basic terms in geometry. We may think of a line as a "straight" line that we might draw with a ruler on a piece of paper, except that in geometry, a line extends forever in both directions. We write the name of a line passing through two different points *A* and *B* as "line AB" or as $\overleftrightarrow{AB}$, the two-headed arrow over *AB* signifying a line passing through points A and B.

Example:
The following is a diagram of two lines: line *AB* and line *HG*.

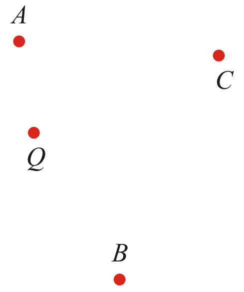


The arrows signify that the lines drawn extend indefinitely in each direction.

Points

A point is another basic term in geometry. We may think of a point as a "dot" on a piece of paper. We denote this point with a letter. A point has no length or width, it just specifies an exact location.

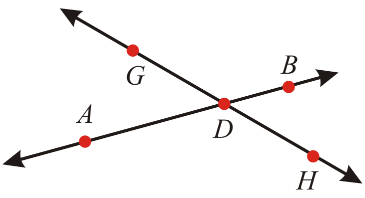
Example:
The following is a diagram of points A, B, C, and Q:



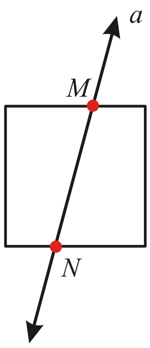
Intersection

The term *intersect* is used when lines, rays, line segments or figures meet, that is, they share a common point. The point they share is called the point of intersection. We say that these figures intersect.

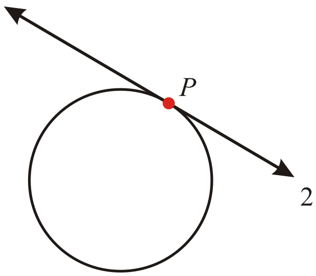
Example:
In the diagram below, line AB and line GH intersect at point D:



Example:
In the diagram below, line a intersects the sides of the square in points M and N:



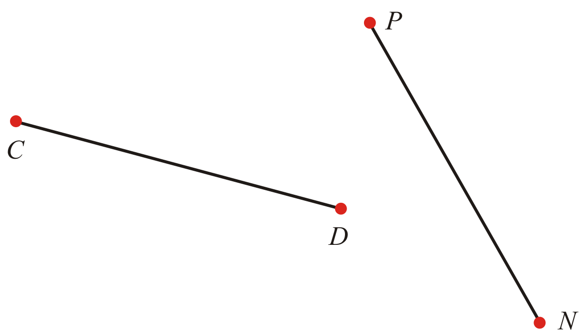
Example:
In the diagram below, line 2 intersects the circle at point *P*:



Line Segments

A line segment is one of the basic terms in geometry. We may think of a line segment as a "straight" line that we might draw with a ruler on a piece of paper. A line segment does not extend forever, but has two distinct endpoints. We write the name of a line segment with endpoints *A* and *B* as "line segment *AB*" or as $\overline{AB}$. Note how there are no arrow heads on the line over *AB* such as when we denote a line or a ray.

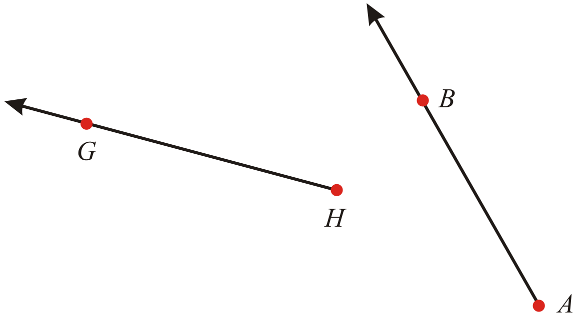
Example:
The following is a diagram of two line segments: line segment *CD* and line segment *PN*, or simply segment *CD* and segment *PN*.



Rays/vectors

A ray is one of the basic terms in geometry. We may think of a ray as a "straight" line that begins at a certain point and extends forever in one direction. The point where the ray begins is known as its endpoint. We write the name of a ray with endpoint *A* and passing through a point *B* as "ray *AB*" or as $\overrightarrow{AB}$. Note how the arrow heads denotes the direction the ray extends in: there is no arrow head over the endpoint.

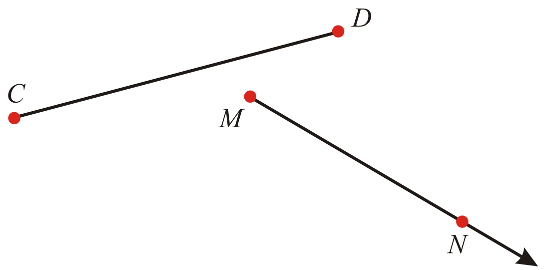
Example:
The following is a diagram of two rays: ray *HG* and ray *AB*.



Endpoints

An endpoint is a point used to define a line segment or ray. A line segment has two end-points, a ray has one.

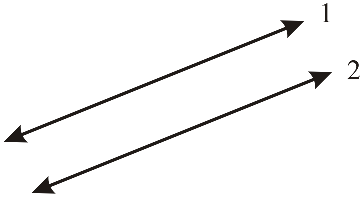
Example:
The endpoints of line segment *DC* below are points *D* and *C*, and the endpoint of ray *MN* is point *M* below:



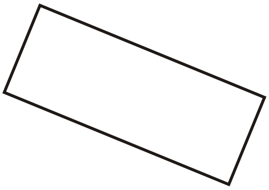
Parallel Lines

Two lines in the same plane which never intersect are called parallel lines. We say that two line segments are parallel if the lines that they lie on are parallel. If line 1 is parallel to line 2, we denote this as line 1 || line 2.
When two line segments *DC* and *AB* lie on parallel lines, we denote this as *DC* || *AB*.

Example:
Lines 1 and 2 below are parallel.

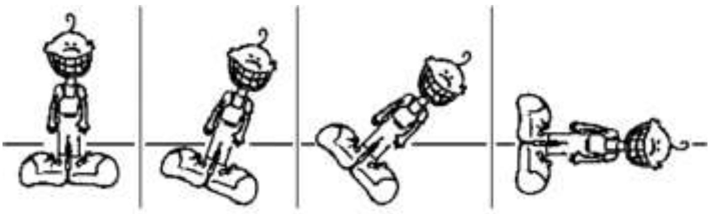


Example:
The *opposite* sides of the rectangle below are parallel. The lines passing through them never meet.

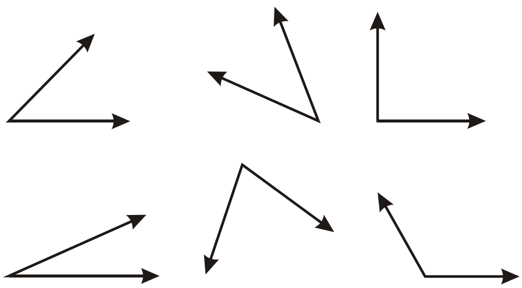


What is an Angle?

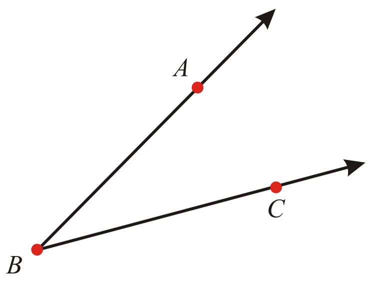
Two rays that share the same initial point form an angle. The point where the rays intersect is called the vertex of the angle. The two rays are called the sides of the angle.



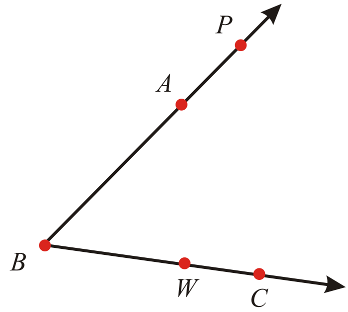
Example:
Here are some examples of angles.



We can specify an angle by using a point on each ray and the vertex. The angle below may be specified as angle *ABC* or as angle *CBA*; you may also see this written as $\angle ABC$ or as $\angle CBA$. Note how the vertex point is always given in the middle.



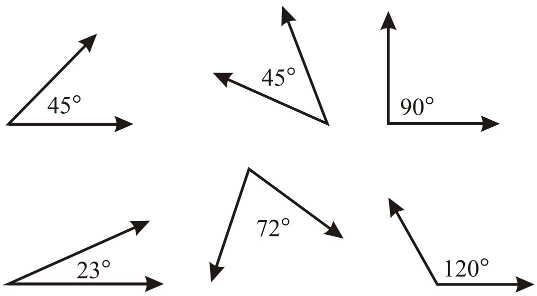
Example:
Many different names exist for the same angle. For the angle below $\angle PBC$, $\angle PBW$, $\angle CBP$, and $\angle WBA$ are all names for the same angle.



Degrees: Measuring Angles

We measure the size of an angle using degrees or radians.

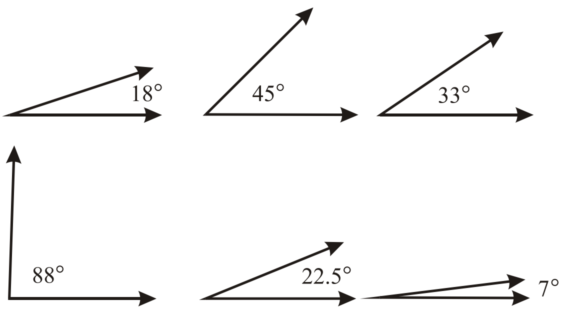
Example:
Here are some examples of angles and their degree measurements.



Acute Angles

An acute angle is an angle measuring between 0 and 90 degrees.

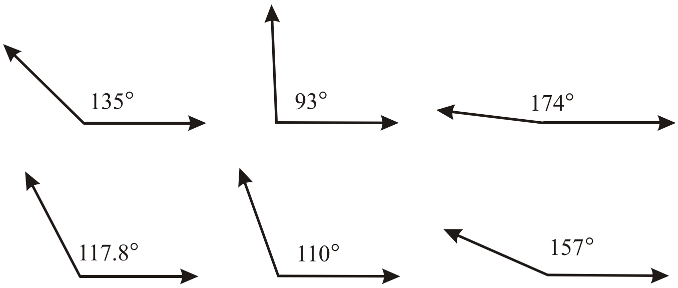
Example:
The following angles are all acute angles.



Obtuse Angles

An obtuse angle is an angle measuring over 90 and less than 180 degrees.

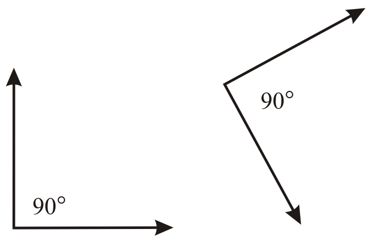
Example:
The following angles are all obtuse.



Right Angles

A right angle is an angle measuring 90 degrees. Two lines or line segments that meet at a right angle are said to be perpendicular. Note that any two right angles are supplementary angles (a right angle is its own angle supplement).

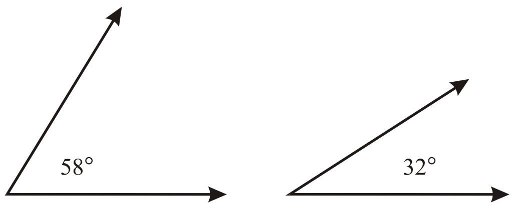
Example:
The following angles are both right angles.



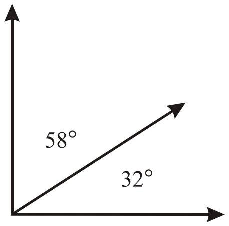
Complementary Angles

Two angles are called complementary angles if the sum of their degree measurements equals 90 degrees. One of the complementary angles is said to be the complement of the other.

Example:
These two angles are complementary.



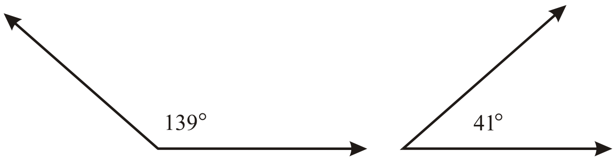
Note that these two angles can be “pasted” together to form a right angle!



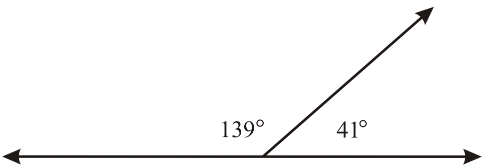
Supplementary Angles

Two angles are called supplementary angles if the sum of their degree measurements equals 180 degrees. One of the supplementary angles is said to be the supplement of the other.

Example:
These two angles are supplementary.

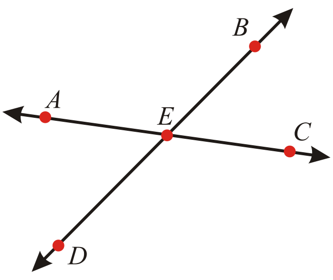


Note that these two angles can be “pasted” together to form a straight line!



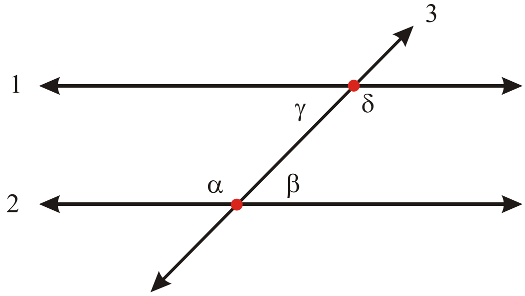
Vertical Angles

For any two lines that meet, such as in the diagram below, angle *AEB* and angle *DEC* are called vertical angles. Vertical angles have the same degree measurement. Angle *BEC* and angle *AED* are also vertical angles.



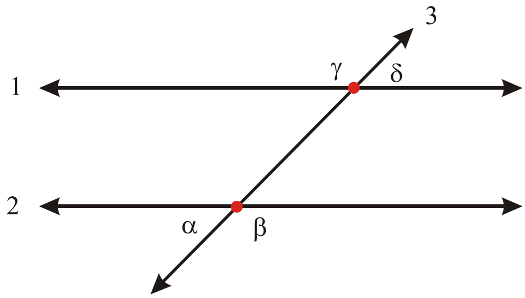
Alternate Interior Angles

For any pair of parallel lines 1 and 2, that are both intersected by a third line, such as line 3 in the diagram below, angle α and angle δ are called alternate interior angles. Alternate interior angles have the same degree measurement. Angle β and angle γ are also alternate interior angles.



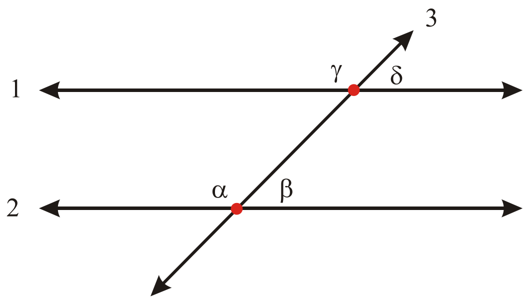
Alternate Exterior Angles

For any pair of parallel lines 1 and 2, that are both intersected by a third line, such as line 3 in the diagram below, angle α and angle δ are called alternate exterior angles. Alternate exterior angles have the same degree measurement. Angle β and angle γ are also alternate exterior angles.



Corresponding Angles

For any pair of parallel lines 1 and 2, that are both intersected by a third line, such as line 3 in the diagram below, angle α and angle γ are called corresponding angles. Corresponding angles have the same degree measurement. Angle β and angle δ are also corresponding angles.

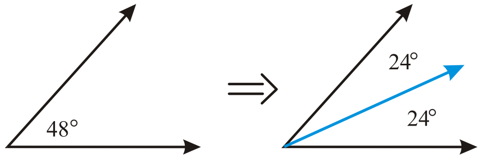


Angle Bisector

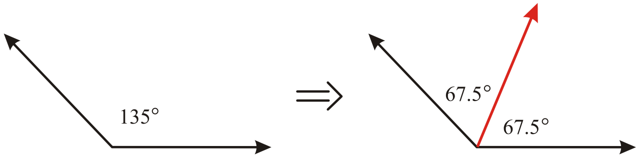
An angle bisector is a ray that divides an angle into two equal angles.

Example:

The blue ray on the right is the angle bisector of the angle on the left.

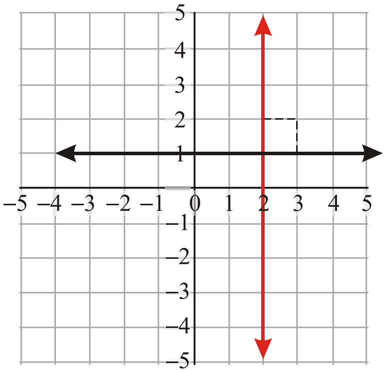


The red ray on the right is the angle bisector of the angle on the left.



Perpendicular Lines

Two lines that meet at a right angle are perpendicular.



Adapted from http://www.mathleague.com

Follow-up exercises

4. Are the following statements true or false?

a) an acute angle measures between 0 and 90 degrees.

b) a right angle measures 180 degrees.

c) two angles are called supplementary if the sum of their degree measurements equals 180 degrees.

d) vertical angles have the same degree measurement.

e) an angle bisector divides an angle into two different angles.

5. Write five definitions of geometrical terms from this chapter. Read them to your partner. Your partner will tell you what the term is. Swap roles.

6. What are the terms for the following definitions?
1. A dot on a plane.
2. When lines, rays, figures share a common point.
3. A straight line that has one endpoint.
4. Two lines in the same plane which never intersect.

7. Draw a picture consisting of various combinations of straight lines and angles. Do not show it to your partner. Work in pairs. Describe the picture to your partner in detail, asking them to draw it. Compare the pictures. Swap roles.

8. Put the following words in the right order to form correct sentences:
a) terms line one the basic is in A geometry of.
b) piece is a dot on of A paper point a.
c) of that share The called lines the point point is intersection.
d) endpoints A does not extend segment forever, but has line two distinct.
e) the where point the begins is ray as its endpoint known.
f) an rays endpoint that the same Two form share angle.
g) degrees We of measure the an angle using size.
h) a 90 right is angle measuring angle degrees an.

Now – try reading aloud:

Examples:
1. The point $P(2,3)$ does not lie on the line $2x - y/4 + 8 = 0$, since $2 \cdot 2 - 3/4 + 8 \neq 0$.
2. The line $x/2 + y/3 - 17 = 0$ does not pass through the origin, since $0/2 + 0/3 - 17 \neq 0$.
3. The point $P_1(57, 88)$ lies on the line $y - 8 = 2 \cdot (x - 17)$, since $88 - 8 = 2 \cdot (57 - 17)$.
4. The line through the points $P_1(0, 3/2)$ and $P_2(2, 5/2)$ has the equation $(y - 3/2) / (x - 0) = (5/2 - 3/2) / 2$ or $y = x/2 + 3/2$. It cuts the x-axis in the point S whose ordinate is $y_0 = 0$. Its abscissa is then $x_0 = -3$. The point $S(-3, 0)$ is in the intersection of the line $y = x/2 + 3/2$ with the x-axis; $x_0 = -3$ is the zero of the function.
5. If a point P_1 with the abscissa $x_1 = 5$ is to lie on the line $y = 2x/3 - 2$, then its ordinate y must have the value $y_1 = 2x_1/3 - 2 = 10/3 - 2 = 4/3$.

---

6. 2 *x* minus *y* divided by 4 plus 8 equals 0
   $2x - y/4 + 8 = 0$.

---

7. 2 minus 3 divided by 4 plus 8 is not equal to 0
   $2 - 3/4 + 8 \neq 0$.

---

8. Point P 1 has coordinates 57 and 88
   $P_1(57, 88)$.

---

9. The line given by the equation *y* minus 8 equals 2 times, open brackets, *x* minus 17, close brackets
   $y - 8 = 2\ (x - 17)$.

☺A mathematician and his best friend, an engineer,
attend a public lecture on geometry in thirteen-dimensional space.
"How did you like it?" the mathematician wants to know after the talk.
"My head's spinning", the engineer confesses.
"How can you develop any intuition for thirteen-dimensional space?"
"Well, it's not even difficult. All I do is visualize the situation
in arbitrary N-dimensional space and then set N = 13."☺

Geometry. Part 2

Lead-in:

1. Warm up: What terms concerning figures in geometry do you know?

2. Listen and repeat.
polygon, *regular polygon*, *vertex*, *triangle*, *equilateral triangle*, *isosceles triangle*, *scalene triangle*, *acute triangle*, *obtuse triangle*, *right triangle*, *quadrilateral*, *rectangle*, *square*, *parallelogram*, *rhombus*, *trapezoid*, *pentagon*, *hexagon*, *heptagon*, *octagon*, *nonagon*, *decagon*, *circle, convex.*

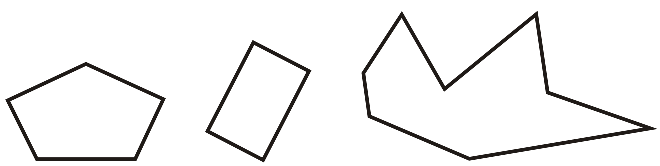
3. Read the text. Find the above words in context.

Text:

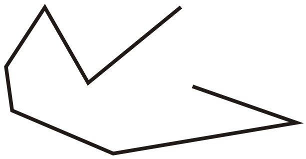
Polygon

A polygon is a closed figure made by joining line segments, where each line segment intersects exactly two others.

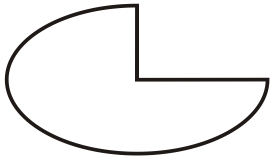
Examples:
The following are examples of polygons:



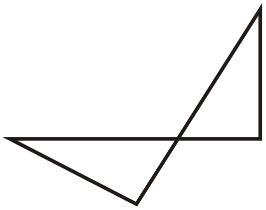
The figure below is not a polygon, since it is not a closed figure:



The figure below is not a polygon, since it is not made of line segments:



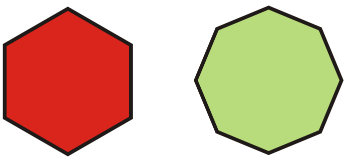
The figure below is not a polygon, since its sides do not intersect in exactly two places each:



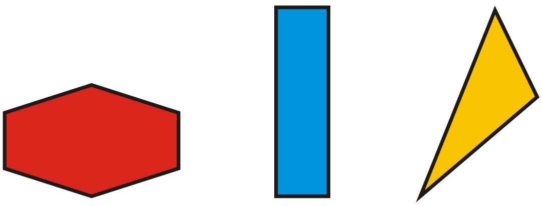
Regular Polygon

A regular polygon is a polygon whose sides are all of the same length, and whose angles are all the same. The sum of the angles of each polygon with n sides, where n is 3 or more, is $180 \times (n - 2)$ degrees.

Examples:
The following are examples of regular polygons:

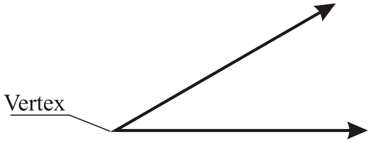


Examples:
The following are examples of non-regular polygons:

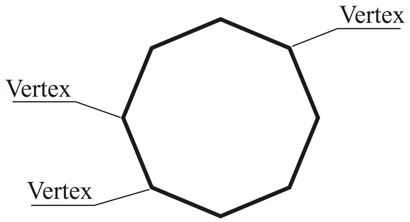


Vertex

1) The vertex of an angle is the point where the two rays that form the angle intersect.



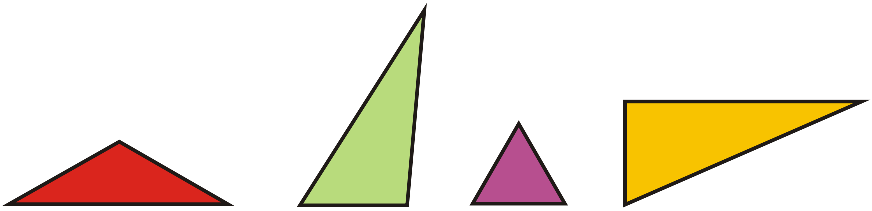
2) The vertices of a polygon are the points where its sides intersect.



Triangle

A three-sided polygon. The sum of the angles of a triangle is 180 degrees.

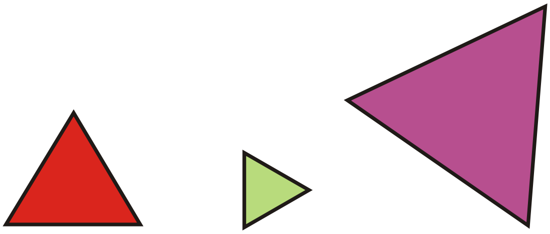
Examples:



Equilateral Triangle or Equiangular Triangle

A triangle having all three sides of equal length. The angles of an equilateral triangle all measure 60 degrees.

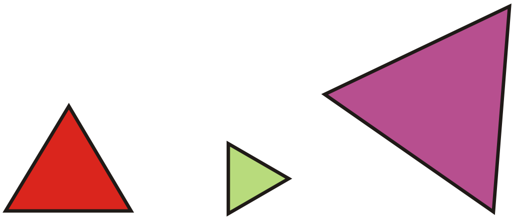
Examples:



Isosceles Triangle

A triangle having at least two sides of equal length.

Examples:

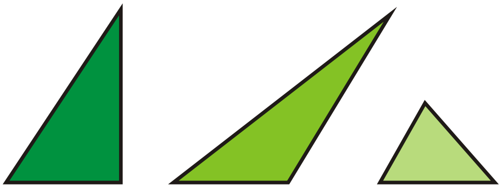


In particular each *equilateral* triangle is an *isosceles* tringle.

Scalene Triangle

A triangle having three sides of different lengths.

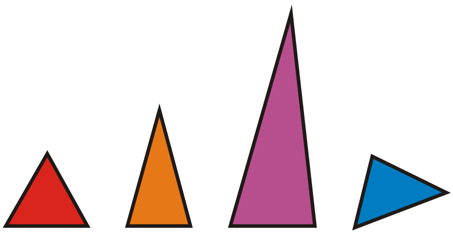
Examples:



Acute Triangle

A triangle having three acute angles.

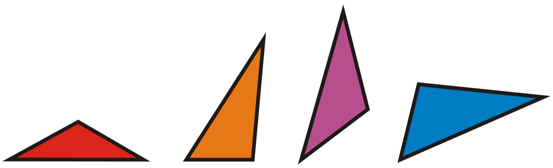
Examples:



Obtuse Triangle

A triangle having an obtuse angle. One of the angles of the triangle measures more than 90 degrees.

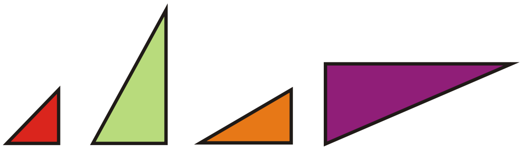
Examples:



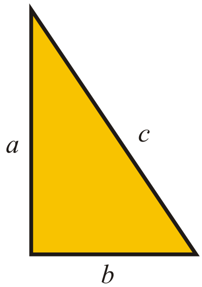
Right Triangle

A triangle having a right angle. One of the angles of the triangle measures 90 degrees. The side opposite the right angle is called the hypotenuse. The two sides that form the right angle are called the legs. A right triangle has the special property that the sum of the squares of the lengths of the legs equals the square of the length of the hypotenuse. This is known as the Pythagorean Theorem.

Examples:

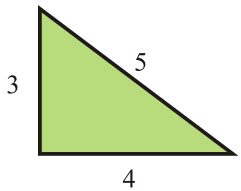


Example:



For the right triangle above, the lengths of the legs are *a* and *b*, and the hypotenuse has length *c*. Using the Pythagorean Theorem, we know that $a^2 + b^2 = c^2$.

Example:



In the right triangle above, the hypotenuse has length 5, and we see that $3^2 + 4^2 = 5^2$ according to the Pythagorean Theorem.

Quadrilateral

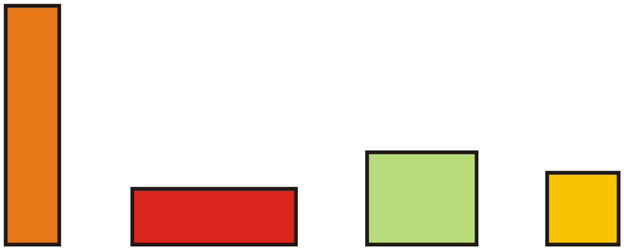
A four-sided polygon. The sum of the angles of a quadrilateral is 360 degrees.

Examples:

Rectangle

A four-sided polygon having all right angles. The sum of the angles of a rectangle is 360 degrees.

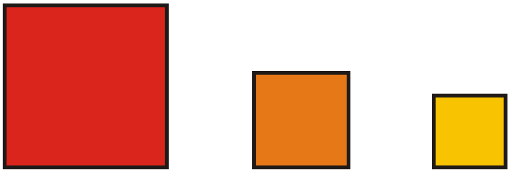
Examples:



Square

A four-sided polygon having equal-length sides meeting at right angles. The sum of the angles of a square is 360 degrees.

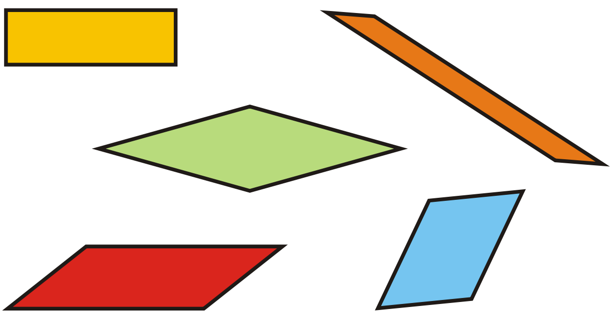
Examples:



Parallelogram

A four-sided polygon with two pairs of parallel sides. The sum of the angles of a parallelogram is 360 degrees.

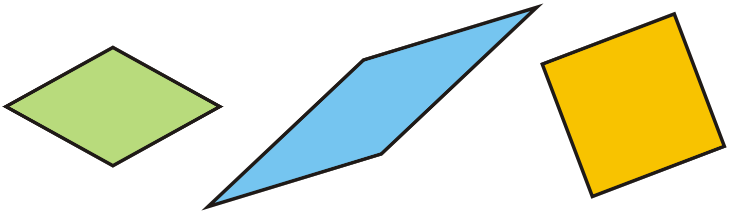
Examples:



Rhombus

A four-sided polygon having all four sides of equal length. The sum of the angles of a rhombus is 360 degrees.

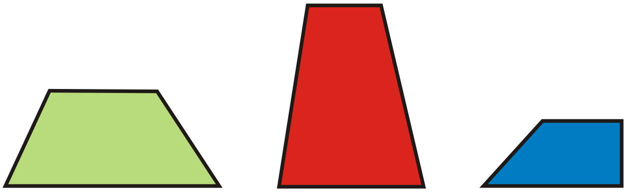
Examples:



Trapezoid

A four-sided polygon having exactly one pair of parallel sides. The two sides that are parallel are called the bases of the trapezoid. The sum of the angles of a trapezoid is 360 degrees.

Examples:

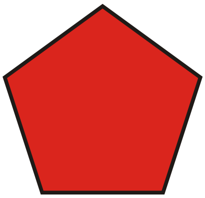


Pentagon

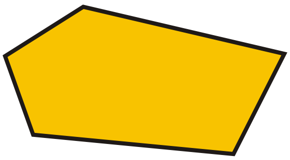
A five-sided polygon. The sum of the angles of a pentagon is 540 degrees.

Examples:

A regular pentagon:



An irregular pentagon:

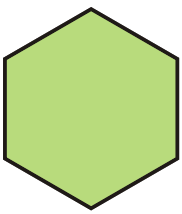


Hexagon

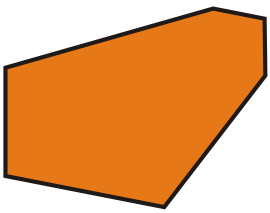
A six-sided polygon. The sum of the angles of a hexagon is 720 degrees.

Examples:

A regular hexagon:



An irregular hexagon:

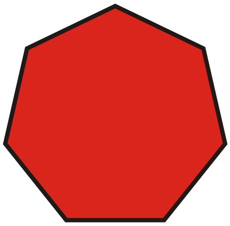


Heptagon

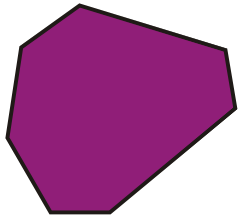
A seven-sided polygon. The sum of the angles of a heptagon is 900 degrees.

Examples:

A regular heptagon:



An irregular heptagon:

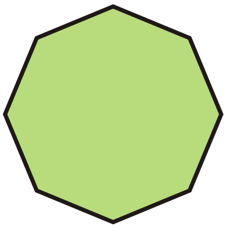


Octagon

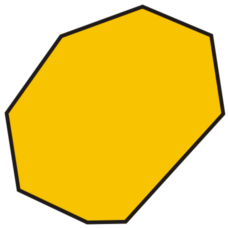
An eight-sided polygon. The sum of the angles of an octagon is 1080 degrees.

Examples:

A regular octagon:



An irregular octagon:

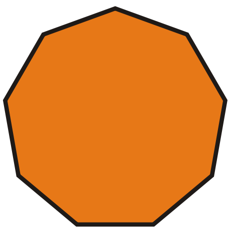


Nonagon

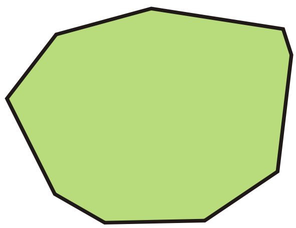
A nine-sided polygon. The sum of the angles of a nonagon is 1260 degrees.

Examples:

A regular nonagon:



An irregular nonagon:

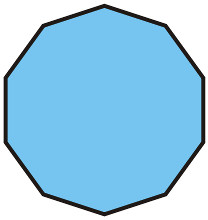


Decagon

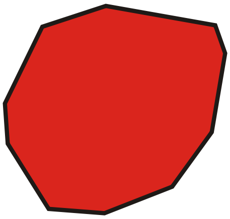
A ten-sided polygon. The sum of the angles of a decagon is 1440 degrees.

Examples:

A regular decagon:



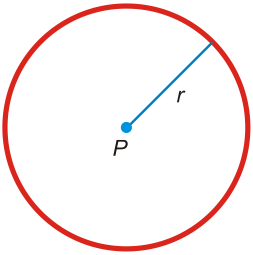
An irregular decagon:



Circle

A circle is the collection of points in a plane that are all at the same distance from a fixed point. The fixed point is called the center. A line segment joining the center to any point on the circle is called a radius.

Example:



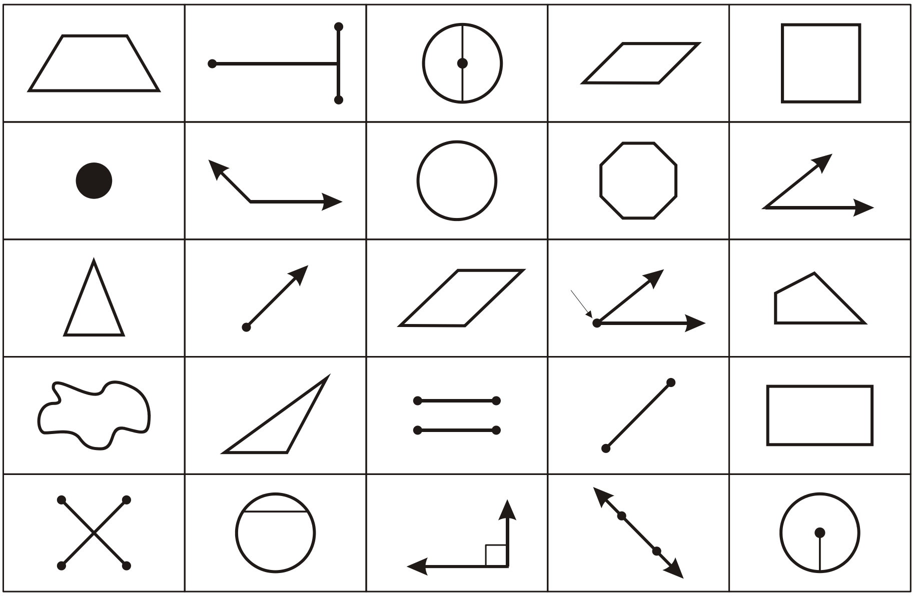
Adapted from http://www.mathleague.com

Follow-up exercises

4. Are the following statements true or false?
   a. A regular polygon has all the sides of the same length and the angles are all the same.
   b. The sum of the angles of a triangle is 360 degrees.
   c. A triangle having two sides of equal length is an equilateral triangle.
   d. A triangle having an obtuse angle is an acute triangle.
   e. The sum of the angles of a rhombus is 180 degrees.
   f. A line segment joining the center to any point in the circle is called a radius.

5. Answer the following questions according to the text:
   a. What is a circle?
   b. What do you call the two sides that are parallel in a trapezoid?
   c. What do you call a four–sided polygon having all right angles?
   d. What do you call the side opposite the right angle in a triangle?
   e. What do you call the two sides that form the right angle in a triangle?
   f. What is a regular polygon?

6. Draw a picture containing 15 different shapes. Work in pairs. Show the picture to your partner. Your partner should find all the shapes and name them in English. Swap roles.
7. Work in pairs. Take turns trying to name the figures in the picture. Then listen to the recording and check:



Now – try reading aloud:

Example:
A cable is to be laid in a straight line through wooded country between two places *R* and *S*. They are not visible from one another, but a point *A* can be found from which the distances $d = |AR| = 2.473$ miles and $e = |AS| = 3.752$ miles and the angle $\tau = \measuredangle RAS = 42°26'10''$ can be measured. What must the length *x* of the cable be and what angles ε, δ from *R*, *S*, respectively must it be laid?
For comparison two methods of solution are given.

***Method* 1**

$x^2 = d^2 + e^2 - 2de \cos \tau = 6.497313$, so $x = 2.549$ miles,
$\sin \varepsilon = (e/x) \sin \tau$. So $\varepsilon_1 = 83°20'00''$ or $\varepsilon_2 = 96°40'00''$ and
$\delta = 180° - (\varepsilon + \tau)$. So $\delta_1 = 54°13'50''$ or $\delta_2 = 40°53'50''$.
Since $e > x > d$, it must also follow that $\varepsilon > \tau > \delta$; this condition is satisfied only by δ_2. Therefore the solution is x, ε_2, δ_2.

***Method* 2**

$$\tan\frac{\varepsilon-\delta}{2} = \frac{e-d}{e+d}\tan\frac{\varepsilon+\delta}{2}$$

$\varepsilon + \delta = 180° - \tau = 137°33'50''$
$(\varepsilon + \delta)/2 = 68°46'55''$

$(\varepsilon - \delta)/2 = 27°53'18''$
So $\varepsilon = 96°40'13''$ and $\delta = 40°53'37''$

$$x = e\frac{\sin\tau}{\sin\varepsilon} = 2.549 \text{ miles.}$$

The agreement between the two results is unsatisfactory. The reason for this is that the sine function was used to determine an angle in the neighbourhood of 90°: sin 96°40′00″ = 0.99324, sin 96°40′10″ = 0.99323, sin 96°40′20″ = 0.99323; on the number of seconds nothing reliable can be said. A greater precision can be obtained in this case if the angle ε is also calculated by cosine rule, that is, from the equation $\cos\varepsilon = (x^2 + d^2 - e^2) / (2xd)$. One obtains the unique value $\cos\varepsilon = -1.464462 / (2 \cdot 2.549 \cdot 2.473)$ or $\varepsilon_2 = 96°40'14''$, in sufficiently close agreement with the value found by the tangent formula. The solution found from the cosine rule is therefore: $x = 2.549$ miles, $\varepsilon_2 = 96°40'14''$, $\delta_2 = 40°53'36''$
d equals the length of segment AR, that is two point four seven three
$d = |AR| = 2.473$.

Angle tau, that is the angle at vertex *A* of triangle *RAS* is 42 degrees 26 minutes and 10 seconds
$\tau = \measuredangle RAS = 42°26'10''$.

☺***Q**: How does a mathematician induce good behavior in her children?*
***A**: "I've told you n times, I've told you n + 1 times…"*☺

Geometry. Part 3

Lead-in:

1. Warm up: Which space figures and basic solids do you know?

2. Listen to the recording and repeat:
depth, *width*, *height*, *cube*, *sphere*, *cylinder*, *prism*, *cone*, *pyramid*, *cross-section*, *volume*, *congruent.*

3. Read the text, write down any other useful terms.

Text:

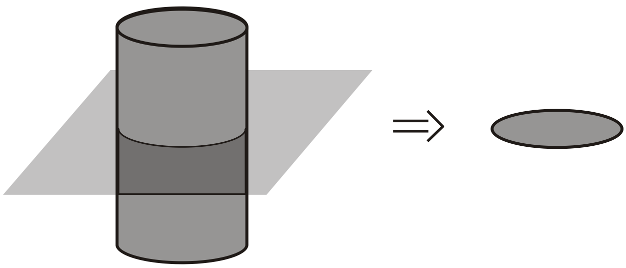
Space Figure (Solid Figure)

A space figure or three-dimensional figure is a figure that has depth in addition to width and height. Everyday objects such as a tennis ball, a box, a bicycle, and a redwood tree are all examples of space figures. Some common simple space figures include cubes, spheres, cylinders, prisms, cones, and pyramids. A space figure having all flat faces is called a polyhedron. A cube and a pyramid are both polyhedrons; a sphere, cylinder, and cone are not.

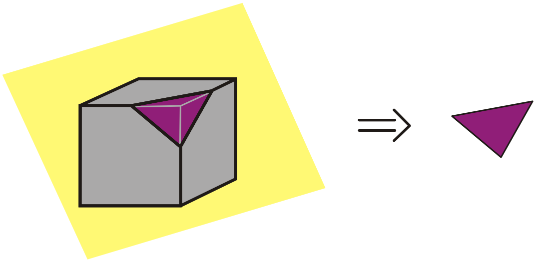
Cross-Section

A cross-section of a space figure is the shape of a particular two-dimensional "slice" of a space figure.

Example:
The circle on the right is an example of a cross-section of the cylinder on the left.



The triangle on the right is an example of a cross-section of the cube on the left.



Volume

Volume is a measure of how much space a space figure takes up. Volume is used to measure a space figure just as area is used to measure a plane figure. The volume of a cube is the cube of the length of one of its sides. The volume of a box is the product of its length, width, and height.

Example:
What is the volume of a cube with side-length 6 cm?

Solution:
The volume of a cube is the cube of its side-length, which is $6^3 = 216$ cubic cm.

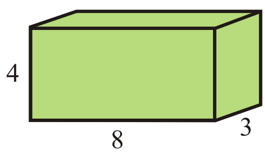
Example:
What is the volume of a box whose length is 4 cm, width is 5 cm, and height is 6 cm?

Solution:
The volume of a box is the product of its length, width, and height, which is $4 \times 5 \times 6 = 120$ cubic cm.

Surface Area

The surface area of a space figure is the total area of all the faces of the figure.

Example:



What is the surface area of a box whose length is 8, width is 3, and height is 4?

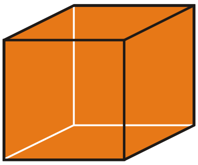
Solution:
This box has 6 faces: two rectangular faces are 8 by 4, two rectangular faces are 4 by 3, and two rectangular faces are 8 by 3. Adding the areas of all these faces, we get the surface area of the box:

$$8 \times 4 + 8 \times 4 + 4 \times 3 + 4 \times 3 + 8 \times 3 + 8 \times 3$$
$$= 32 + 32 + 12 + 12 + 24 + 24 =$$
$$= 136.$$

Cube

A cube is a three-dimensional figure having six matching square sides. If L is the length of one of its sides, the volume of the cube is $L^3 = L \times L \times L$. A cube has six square-shaped sides. The surface area of a cube is six times the area of one of these sides.

Example:
The space figure pictured below is a cube. The grayed lines are edges hidden from view.



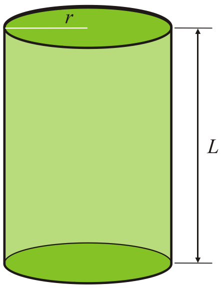
Example:
What is the volume and surface of a cube having a side-length of 2.1 cm?

Solution:
Its volume is 2.1 × 2.1 × 2.1 = 9.261 cubic centimeters.
Its surface area is 6 × 2.1 × 2.1 = 26.46 square centimeters.

Cylinder

A cylinder is a space figure having two congruent (having the same shape and size) circular bases that are parallel. If L is the length of a cylinder, and r is the radius of one of the bases of a cylinder, then the volume of the cylinder is $L\pi r^2$, and the surface area is $2 \times r \times \pi \times L + 2 \times \pi \times r^2$.

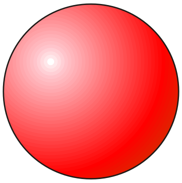
Example:
The figure pictured below is a cylinder. The grayed lines are edges hidden from view.



Sphere

A sphere is a space figure having all of its points at the same distance from its center. The distance from the center to the surface of the sphere is called its radius. Any cross-section of a sphere is a circle.
If r is the radius of a sphere, the volume V of the sphere (ball) is given by the formula $V = 4/3\pi r^3$.
The surface area S of the sphere is given by the formula $S = 4\pi r^2$.

Example:
The space figure pictured below is a sphere.



Example:
To the nearest tenth, what is the volume and surface area of a sphere having a radius of 4 cm?

Solution:
Using an estimate of 3.14 for π, the volume is
$4/3 \times 3.14 \times 4^3 = 4/3 \times 3.14 \times 4 \times 4 \times 4 =268$ cubic centimeters.
Using an estimate of 3.14 for π, the surface area would be
$4 \times 3.14 \times 4^2 = 4 \times 3.14 \times 4 \times 4 = 201$ square centimeters.

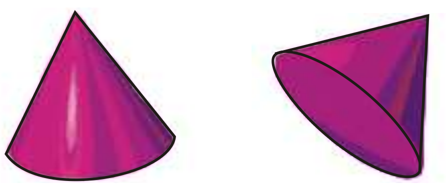
Cone

A cone is a space figure having a circular base and a single vertex.
If r is the radius of the circular base, and h is the height of the cone, then the volume of the cone is $1/3 \times \pi \times r^2 \times h$.

Example:
What is the volume in cubic cm of a cone whose base has a radius of 3 cm, and whose height is 6 cm, to the nearest tenth?

Solution:
We will use an estimate of 3.14 for π.
The volume is $1/3 \times \pi \times 3^2 \times 6 = \pi \times 18 = 56.52$, which equals 56.5 cubic cm when rounded to the nearest tenth.

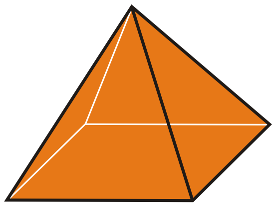
Example:
The pictures below presents two different views of a cone.



Pyramid

A pyramid is a space figure with a square base and 4 triangle-shaped sides.

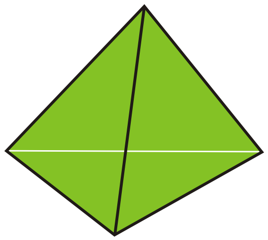
Example:
The picture below shows a pyramid. The grayed lines are edges hidden from view.



Tetrahedron

A tetrahedron is a 4-sided space figure. Each face of a tetrahedron is a triangle.

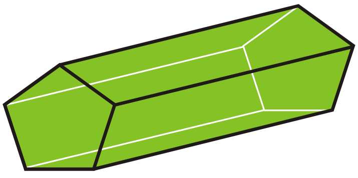
Example:
The picture below presents a tetrahedron. The grayed line is the edge hidden from view.



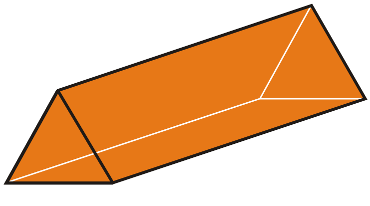
Prism

A prism is a space figure with two congruent, parallel bases that are polygons.

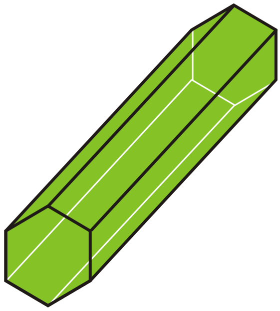
Examples:
The figure below shows a pentagonal prism (the bases are pentagons). The grayed lines are edges hidden from view.



The figure below presents a triangular prism (the bases are triangles). The grayed lines are edges hidden from view.



The figure below depicts a hexagonal prism (the bases are hexagons). The grayed lines are edges hidden from view.



Adapted from http://www.mathleague.com

Follow-up exercises

4. Answer the following questions:
   a. How many sides does a tetrahedron have?
   b. How many vertices does a cone have?
   c. How many vertices does a sphere have?
   d. How do you call the distance from the center to the surface of the sphere?
   e. How many bases does a cylinder have?
   f. What is a space figure?

5. Work in groups of three. Prepare six true/false sentences about the text. Then change groups. One of you should read your sentences and the group should decide if they are true or false.

6. Each of you should draw a picture containing 15 geometrical figures. Then show your picture to your partner. The partner should find and identify (name) the figures. Swap roles.

7. Give the plural forms of the following words:

depth	height
sphere	ball
cross-section	cone
slice	box
surface	radius

8. Put the following words in the right order, to form correct sentences:
   a) congruent, prism a bases space figure is with A two parallel.
   b) from The radius. distance the to the sphere surface of the is center called its
   c) a three cube is sides a -dimensional having six figure matching square.
   d) a depth has space figure length in addition width and to.
   e) all A space sides polyhedron faces figure having flat is called a faces
   f) measure Volume figure is a how space of a much space takes up.

9. Work in pairs. Decide who is A, and who is B. Person A – go to Appendix 2A, person B – go to Appendix 2B. Read your text, ask your partner questions to find out the missing information in your text. Swap roles.

Now – try reading aloud:

***Example* 1**

a) to find the surface area of a regular 6-sided prism with base-edge $a = 3u$ and $h = 4u$, where u is the unit of length.

b) $B = 3/2\ a^2\ \sqrt{3}$ and $C = 6ah$, and so $S = 2 \cdot 3/2\ a^2\ \sqrt{3} + 6ah = 3a^2\ \sqrt{3} + 6ah = 3a(a\ \sqrt{3} + 2h)$.

c) by substituting the given values one obtains $S \approx 119u^2$ (square units, say square inches).

***Example* 2**

For the surface area of a steel bolt of circular cross-section with diameter $d = 50u$ and height $h = 60u$, one obtains $B = \pi d^2/4$ and $C = \pi dh$, and so $S = 2\ \pi d^2/4 + \pi dh = \pi d(d/2 + h)$ and, by substitution the given values, $S = 2450\pi u^2 \approx 13352u^2$.

---

1. *V* equals 4 divided by 3 times pi times r cubed (*V* is the volume of a sphere with radius *r*)
   $V = 4/3\pi r^3$.

---

2. S equals 4 times pi times *r* squared (*S* is the surface of the sphere with radius *r*)
   $S = 4\pi r^2$.

☺ Three statisticians go hunting. When they see a rabbit, the first one shoots, missing it on the left. The second one shoots and misses it on the right. The third one shouts: "We've hit it!"☺

Probability

Lead-in:

1. Warm up: listen to your teacher. (A group exercise).

2. Repeat after the recording:
die, *to throw the dice*, *matches*, *independent*, *related*, *mutually*, *exclusive*, *outcome*, *occur*, *to bear something in mind*, *let's assume*, *assumption*, *multiply*, *event*, *elementary event*, *identical event.*

3. Read the text. Which examples of the theory of probability are described in the text?

Text:

The Birthday Problem

The birthday problem is a famous way of finding the probability of two people in one room having the same birthday. Let's consider the situation of one person: that person has a chance of 365 days as his or her birthday, naturally ignoring the issue of leap years.
Considering the situation of two people, the second person has a possibility of having birthday on 364 days that would be different from the first person's birthday.
If we consider three people, the third person has 363 different days that could be his or her birthday and not shared with the other two people. Therefore the probability is as follows:

$$1-(365\cdot364\cdot363)/(365\cdot365\cdot365) = 0.0082$$

In order to obtain this you must figure out the probability that no one in the group has a match. To do this just find the product of each distinct day over the total days.

$$365/365\cdot364/365\cdot363/365 = 0.9918$$

Next, to find out the probability that at least two people have the same birthday do the following operation:

$1 – 0.9918 = 0.0082$

From the operation above we can extract an equation that will help us find such probabilities in bigger groups.

$P(A) = 365·364·363 ·...· (365 – N + 1)/365^N$
$P(AC) = 1 – 365 · 364 · 363 ·...· (365 – N + 1)/365^N$

Where: N = number of people in the group, P(A) = Probability that no one shares a birthday, P(AC) = Probability that at least 2 people share the same birthday.

An event in probability theory is any fact, which may occur as a result of an experiment with a random outcome or may not. The simplest result of such experiment is called ***an elementary event*** (for instance, an appearance of heads or tails at throwing of a coin, shooting hit, an appearance of an ace at taking a card out of a pack, a random appearance of number at throwing of a die etc.).

A set of all elementary events *E* is called ***a space of elementary events***. So, this space consists of six elementary events at throwing of a die and 52 elementary events at taking a card out of a pack. An event can consist of one or several elementary events, for example, an appearance of two aces one after the other at taking a card out of a pack, or an appearance of the same number at triple throwing of a die. Then it's possible to define ***an event*** as an arbitrary subset of a space of elementary events.

A certain event is the whole space of elementary events. Thus, ***a certain event*** is an event, which must happen as a result of the experiment without fail. Such event at throwing of a die is a fall of the die on one of its faces.

An impossible event ($\varnothing$) is called an empty subset of a space of elementary events. That is, ***an impossible event*** cannot happen as a result of the experiment. So, such an event at throwing of a die is a fall of the die on its edge.

Events *A* and *B* are called ***identical events*** ($A = B$), if the event *A* occurs if and only if the event *B* occurs. An event *A* ***involves the event*** *B* ($A \subset B$), if the condition "*the event B occurred*" follows from the condition "*the event A occurred*".

An event *C* is called ***a sum of the events*** *A* and *B* ($C = A \cup B$), if the event *C* happens if and only if either the event *A* happens or the event *B* happens.
An event *C* is called ***a product of the events*** *A* and *B* ($C = A \cap B$), if the event *C* happens if and only if both event *A* and event *B* happen.
An event *C* is called ***a difference of the events*** *A* and *B* ($C = A – B$), if the event *C* happens if and only if the event *A* happens and the event *B* doesn't.

An event A' is ***a complementary event*** to the event *A*, if the event *A* doesn't occur. So, shooting hit and miss are complementary events.

Independent or mutually exclusive events?

One of the important steps you need to make when considering the probability of two or more events occurring is to decide whether they are independent or related events.

Mutually Exclusive vs. Independent

It is not uncommon for people to confuse the concepts of mutually exclusive events and independent events.

Definition of mutually exclusive events

If event A happens, then event B cannot, or vice-versa. The two events "it rained on Tuesday" and "it did not rain on Tuesday" are mutually exclusive events. When calculating the probability of the union of exclusive events you add the probabilities.

Independent events

The outcome of event *A*, has no effect on the outcome of event *B*. Such as "It rained on Tuesday" and "My chair broke at work". When calculating the probabilities for independent events you multiply the probabilities. You are effectively saying what is the chance of both events happening, bearing in mind that the two were unrelated.
So, if two nonempty events *A* and *B* are mutually exclusive, they cannot be independent. If *A* and *B* are independent, they cannot be mutually exclusive. However, if the events were "it rained today" and "I left my umbrella at home" they are not mutually exclusive, but they are probably not independent either, because one would think that you'd be less likely to leave your umbrella at home on days when it rains. That fact aside use the following to understand the definition.

Example of mutually exclusive events

What happens if we want to throw 1 and 6 in any order? This now means that we do not mind if the first die is either 1 or 6, as we are still in with a chance. But with the first die, if 1 falls uppermost, clearly it rules out the possibility of 6 being uppermost, so the two outcomes, 1 and 6, are exclusive. One result directly affects the other. In this case, the probability of throwing 1 or 6 with the first die is the sum of the two probabilities, 1/6 + 1/6 = 1/3.
The probability of the second die being favourable is still 1/6 as the second die can only be one specific number, a 6 if the first die is 1, and vice versa.
Therefore the probability of throwing 1 and 6 in any order with two dice is 1/3 × 1/6 = 1/18.
Note that we multiplied the last two probabilities as they were independent of each other.

An example of independent events

The probability of throwing a double three with two dice is the result of throwing three with the first die and three with the second die. The total possibilities are, one from six outcomes for the first event and one from six outcomes for the second, Therefore (1/6) × (1/6) = 1/36 or 2.77%.
The two events are independent, since whatever happens to the first die cannot affect the throw of the second, the probabilities are therefore multiplied, and remain 1/36th.

Converse (complementary) probabilities

Converse – "Something that has been reversed; an opposite"
When you work out the probability of an event, you do not need to work out the probability of an event occurring, it's enough to know the opposite. The probability that the event will

not occur. For example, the probability of throwing a 1 on a die is 1/6 therefore the probability of a ‘non-1’ is 1–1/6 which equals 5/6.
Converse probabilities are used to work out such problems as: “What is the probability of exactly one soccer match ending in a draw within a group of three separate matches?”
Let us assume the chance of a draw occurring in any match is 1/3 or 33.33%. To fulfill our target of only one match ending in a draw we would require the other matches to not end in a draw or (1–(1/3)) which equals 2/3 or 66.66%.
Therefore the probability of only one match out of three being drawn is 1/3 × 2/3 × 2/3 which equals = 4/27 or (0.33 × 0.67 × 0.67) = 14.81%
In our group of three matches there are three ways for only one match to draw, *DXX*, *XDX*, *XXD*, therefore we need to add together all the probabilities, three in this case.
The final answer to the probability of one match drawing is (4/27) + (4/27) + (4/27) = 4/9 or 0.1481 + 0.1481 + 0.1481 = 44.44%.

Adapted from www.peterweb.co.uk

Follow-up exercises

4. What examples of application of the probability theory can you think of?

5. In pairs think of an example of application of the probability theory, describe it and then present to the group.

6. Write 6 true/false sentences referring to the text and quiz each other in pairs or groups of three.

7. Make a list of 10 useful terms from this chapter. Quiz other students in groups of three.

8. Read the problem carefully, explain the difficult words. In groups of three try to find the possible solutions.

Expected value has very practical applications. For example, it can be used in the study of infectious diseases. The following is an extremely simplified version of such a study. Despite the somewhat unrealistic nature of the problem, it should help you to see how this statistic can be used.

Six (unusually sociable) hermits live on an otherwise deserted island. An infectious disease strikes the island. The disease has a 1-day infectious period and after that the person is immune (cannot get the disease again). Assume one of the hermits gets the disease (maybe from a piece of the Mir space station). He randomly visits one of the other hermits during his infectious period. If the visited hermit has not had the disease, he gets it and is infectious the following day. The visited hermit then visits another hermit. The disease is transmitted until an infectious hermit visits an immune hermit, and the disease dies out. There is one hermit visit per day. Assuming this pattern of behaviour, how many hermits can be expected, on the average, to get the disease?

Adapted from: www.mathforum.org

Questions:

What is the least number of hermits that could get infected?
What is the greatest number of hermits that could get infected?

What sort of model could you use for this problem?
How would you solve this problem analytically?
How would changing the number of hermits on the island affect the expected number of infected hermits?

Now – try reading aloud:

***Example*:**
In the game of *Bingo* k = 5 different numbers can be chosen from n = 90 numbers in $\binom{90}{5}$ = 43 949 268 ways. Only if one has cards with all these possibilities is one certain to have a line of five. The number of fours and threes can be calculated in a similar way. From the five (correct) numbers one is always missing; thus, there are $\binom{5}{4}$ = 5 combinations of four. For three correct numbers, with two always missing, there are $\binom{5}{3}$ = 10 combinations. Each of the five combinations of four occurs $\binom{90-5}{1}$ = 85 times among possible incorrect lines of five, since this is the number of lines with four correct numbers is $\binom{5}{4}\binom{90-5}{1}=\binom{5}{4}\binom{85}{1}=5\cdot 85=425$. Each combination of three numbers can be combined with two of the remaining numbers in $\binom{85}{2}=85\cdot 42$ ways so that neither a line of four correct nor one of five correct results. Thus, the number of lines with three correct numbers is $\binom{5}{2}\binom{85}{2}$ = 35 700.

---

1. The probability of event A equals 365 times 364 times … times 365 minus N plus 1 all divided by 365 to N-th power
   $P(A)$ = (365)*(364)*(363)*…(365 – N + 1)/365^N

---

2. The binomial coefficient 90 over 5 equals 43 million 949 thousand 268
   $\binom{90}{5}$ = 43 949 268

---

3. 5 over 4 times 90 – 5 over 1 equals 5 over 4 times 85 over 1 equals 5 times 85 equals 425
   $\binom{5}{4}\binom{90-5}{1}=\binom{5}{4}\binom{85}{1}=5\cdot 85=425$.

☺ Statistics Canada is hiring mathematicians.
Three recent graduates are invited for an interview: one has a degree in pure mathematics, another one in applied math, and the third one obtained his B.Sc. in statistics.
All three are asked the same question: "What is one third plus two thirds?"
The pure mathematician: "It's one."
The applied mathematician takes out his pocket calculator, punches in the numbers, and replies: "It's 0.999999999."
The statistician: "What do you want it to be?"☺

Combinatorics

Lead-in:

1. Discuss the definition of combinatorics with your partner.

2. Repeat after the recording:
set, factorial, square, collection, subset, combination, permutation, graph, enumeration, design, discrete, complement set, disjoint set, a deck of playing cards, boundary, principal, significant, to concern, retrieval.

3. Choose five of these terms and discuss their meaning in pairs.

4. Read the text and find these words in the text:

Text:

What is Combinatorics?

Combinatorics is the mathematics of discretely structured problems. Although its boundaries are not easily defined, combinatorics includes the theories of graphs, enumeration, and designs. It is a subject which in the past was studied principally for its aesthetic appeal. Today's modern technology with its vital concern for the discrete has given combinatorics new challenges and a new seriousness of purpose. In particular, since computers require discrete formulations of problems, combinatorics has become indispensable to modern computer science.

An example of a combinatorial question is the following: What is the number of possible orderings of a deck of 52 playing cards? That number equals 52! (i.e., fifty-two factorial). It may seem surprising that this number, about $8.065817517094 \times 10^{67}$, is so large – a little bit more than 8 followed by 67 zeros! Comparing that number to some other large numbers, it is greater than the square of Avogadro's number, 6.022×10^{23}.

Major Areas of Combinatorics

Enumeration

Enumeration is concerned with determining the number of structures with prescribed properties, and is a frequently used tool in mathematics. Many enumeration problems arise from ranking and significance testing in statistics, from probability theory, from telephone networks and from mathematics itself. Extensive calculations are often necessary, and the field of computational enumeration is an important one. The task of getting a computer to evaluate formulas obtained by applying enumerative methods is one requiring considerable experience both with computers and with combinatorics.

Combinatorial Designs

The study of designs deals with a very important and central problem of combinatorial theory, that of arranging objects into patterns according to specified rules. The principal mathematical tools employed are graph theory, number theory, linear and abstract algebra. The construction of magic squares is an example of a design problem, and before the start of the nineteenth century such problems were mainly of recreational interest. In more recent times the concept of a geometry involving only finitely many points has been developed, generalized, and built into the elegant and useful theory of combinatorial designs. Today this area embodies the mathematical tools of such applied areas as the design of experiments, tournament scheduling, information retrieval and coding theory.

Graph Theory

Based upon the simple idea of points interconnected by lines, graph theory combines these basic ingredients into a rich and useful theory, which provides powerful tools for constructing models and solving problems concerning discrete arrangements of objects. Technology today poses a great number of problems that require the construction of complex systems through specific arrangements of their components. These include problems in the scheduling of industrial processes, communication systems, electrical networks, organic-chemical identification, economics and numerous other applied areas.

Enumerative combinatorics

Calculating the number of ways that certain patterns can be formed is the beginning of combinatorics. Let S be a set with n objects. Combinations of k objects from this set S are subsets of S having k elements each (where the order of listing the elements does not distinguish two subsets). Permutations of k objects from this set S refer to sequences of k different elements of S (where two sequences are considered different if they contain the same elements but in a different order, or if they have a different length). Formulas for the number of permutations and combinations are readily available and important throughout combinatorics.

Extremal combinatorics

Many extremal questions deal with set systems. A simple example is the following: what is the largest number of subsets of an *n*-element set one can have, if no two of the subsets are disjoint?
Answer: half the total number of subsets.
Proof: Call the n-element set S. Between any subset T and its complement $S - T$, at most one can be chosen. This proves the maximum number of chosen subsets is not greater than half the number of subsets. To show one can attain half the number, pick one element x of S and choose all the subsets that contain x.
A more difficult problem is to characterize the extremal solutions; in this case, to show that no other choice of subsets can attain the maximum number while satisfying the requirement.
Often it is too hard even to find the extremal answer $f(n)$ exactly and one can only give an asymptotic estimate.

Adapted from http://www.werner-heise.de/combinatorics.html

Follow-up exercises

5. Write short definitions of the following terms:
set, *graph*, *disjoint subsets*, *complement subsets*, *combinatorial design*.

Read your definition to your partners and ask them to guess the words.

6. Do not look back at the text. Are the following sentences true or false?
  a. The boundaries of combinatorics are strictly defined.
  b. Computers do not require discrete formulations.
  c. Enumeration is concerned with determining the number of structures with prescribed properties.
  d. The objects in the collections have to satisfy specified criteria.
  e. Information retrieval is not connected with combinatorics.
  f. Formulas for the number of permutations and combinations are unimportant throughout combinatorics.

7. Write 5 questions to the text. Quiz your partner.

8. Work in pairs. Decide who is *A* (go to Appendix 4A), and who is *B* (go to Appendix 4B). Read your text carefully; prepare questions about the gaps in your text. Work in pairs – ask your partner your questions to get the missing information. Swap roles.

Now – try reading aloud:

By means of *Braille* the blind are able to feel letters, numbers and punctuation, which are printed on paper as arrangements of six points that appear either raised or non-raised. *Point* and *no point* are the two variable elements, and the number of possible signs represented by them is the number of permutations with repetition of these two elements taken 6 at a time. This gives $2^6 = 64$ signs. These possibilities are enough to represent the blind alphabet, together with numbers and punctuation.

Chapter 11

☺The math professor just accepted a new position at a university in another city and has to move. He and his wife pack all their belongings into cardboard boxes and have them shipped off to their new home. To sort out some family matters, the wife stays behind for a few more days while her husband has already left for their new residence. The boxes arrive when the wife still hasn't rejoined her husband. When they talk on the phone in the evening, she asks him to count the boxes, just to make sure the movers didn't lose any of them.
"Thirty nine boxes altogether", says the prof on the phone.
"That can't be", the wife exclaims.
"The movers picked up forty boxes at our old place."
The prof counts once again, but again his count only reaches 39. The next morning, the wife calls the moving company and complains. The company promises to check; a few hours later, someone calls back and reports that all forty boxes did arrive. In the evening, when the prof and his wife are on the phone again, she asks: "I don't understand it. When you count, you get 39, and when they do, they get 40. That's more than strange..."
"Well", the prof says. "This is a cordless phone, so you can stay on the line and count with me: zero, one, two, three, ..."☺

Differential Equations

Lead-in:

1. What is a differential equation?

2. What applications of differential equations do you know?

3. Listen and repeat after the recording:
equation, derivative, function, variable, relativity, coefficient, dynamics, ordinary, linear, solution, boundary, value, homogenous.

4. Read the text and find other useful mathematical terms.

Text:

In mathematics, a **differential equation** is an equation in which the derivatives of a function appear as variables. Many of the fundamental laws of physics, chemistry, biology and economics can be formulated as differential equations. The mathematical theory of differential equations has developed together with the fields of science where the equations originate and where the results find application. Diverse scientific fields often give rise to identical problems in differential equations. In such cases, the mathematical theory can unify otherwise quite distinct scientific fields.

Famous differential equations

- Maxwell's equations in electromagnetism
- Einstein's field equation in general relativity
- The Schrödinger equation in quantum mechanics
- The heat equation in thermodynamics
- The wave equation
- The geodesic equation
- Laplace's equation, which defines harmonic functions
- Poisson's equation
- The Navier-Stokes equations in fluid dynamics
- The Lotka-Volterra equation in population dynamics
- The Black-Scholes equation in finance
- The Cauchy-Riemann equations in complex analysis

Ordinary Differential Equations

Equation: Equations describe the relations between the dependent and independent variables. An equality sign "=" is required in every equation.

Differential Equation: Equations that involve dependent variables and their derivatives with respect to the independent variables are called differential equations.

Ordinary Differential Equation: Differential equations that involve only one independent variable are called ordinary differential equations.

Partial Differential Equation: Differential equations that involve two or more independent variables are called partial differential equations.

Order and Degree

Order: The order of a differential equation is the highest derivative that appears in the differential equation.

Degree: The degree of a differential equation is the power of the highest derivative term.

Linear, Non-linear, and Quasi-linear

Linear: A differential equation is called linear if there are no multiplications among dependent variables and their derivatives. In other words, all coefficients are functions of independent variables.

Non-linear: Differential equations that do not satisfy the definition of linear are non-linear.

Quasi-linear: For a non-linear differential equation, if there are no multiplications among all dependent variables and their derivatives in the highest derivative term, the differential equation is considered to be quasi-linear.

Homogeneous

Homogeneous: A differential equation is homogeneous if every single term contains the dependent variables or their derivatives.

Non-homogeneous: Differential equations which do not satisfy the definition of homogeneous are considered to be non-homogeneous.

Solutions

General Solution: Solutions obtained from integrating the differential equations are called general solutions. The general solution of an ordinary differential equation contains arbitrary constants resulting from integrating times.

Particular Solution: Particular solutions are the solutions obtained by assigning specific values to the arbitrary constants in the general solutions.

Singular Solutions: Solutions that can not be expressed by the general solutions are called singular solutions.

Conditions

Initial Condition: Constrains that are specified at the initial point, generally time point, are called initial conditions. Problems with specified initial conditions are called initial value problems.

Boundary Condition: Constrains that are specified at the boundary points, generally space points, are called boundary conditions. Problems with specified boundary conditions are called boundary value problems.

Adapted from http://www.efundia.com/math/ode.generalterms

Follow-up exercises

5. Write six true/false sentences to the text. Read your sentences in groups of three. Decide if they are true or false.

6. Read the text below and answer the following questions:
   a. What is the order of the differential equation?
   b. What is a linear differential equation?
   c. What operations are accepted for the variable y?
   d. What kind of problem is called an "initial value problem"?
   e. What is the "general solution"?

A **differential equation** is an equation involving an unknown function and its derivatives.
The **order** of the differential equation is the order of the highest derivative of the unknown function involved in the equation.
A **linear differential equation** of order *n* is a differential equation written in the following form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

where $a_n(x)$ is not the zero function. Note that some may use the notation y', y'', y''', $y^{(4)}$, … for the derivatives.
A linear equation obliges the unknown function y to have some restrictions. Indeed, the only operations which are accepted for the variable y are:

(i) differentiating y;
(ii) multiplying y and its derivatives by a function of the variable x;
(iii) adding what you obtained in (ii) and let it be equal to a function of x.

Existence:
Does a differential equation have a solution?

Uniqueness: Does a differential equation have more than one solution? If so, how can we find a solution which satisfies particular conditions?

A problem in which we are looking for the unknown function of a differential equation where the values of the unknown function and its derivatives at some point are known is called an **initial value problem** (in short IVP).

If no initial conditions are given, we call the description of all solutions to the differential equation the **general solution**.

7. Cover the text. Ask the questions in pairs and try to answer correctly.

8. Try to read the following equations in pairs.

The following are examples of important ordinary differential equations which commonly arise in problems of mathematical physics.

Abel's differential equation

a) $y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3 + \ldots$

b) $[g_0(x) + g_1(x)y]y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3$

a) derivative of y equals function f_0 of x plus f_1 times y plus f_2 times y squared plus f_3 times y cubed, and so on

b) the sum of functions g_0 and g_1 of x times the derivative of y equals f_0 of x plus f_1 times y plus f_2 times y squared plus f_3 times y cubed.

Airy differential equation

$$y'' - xy = 0$$

second derivative of y minus x times y equals zero

Anger differential equation

$$y'' + \frac{y'}{x} + \left(1 - \frac{v^2}{x^2}\right)y = \frac{x - v}{\pi x^2}\sin(vx)$$

second derivative of y plus first derivative of y divided by x plus product of one minus ni squared divided by x squared and y equals x minus nu all divided by pi times x squared that all multiplied by sine of nu times x

Baer differential equations

$$(x-\alpha_1)(x-\alpha_2)y''+\frac{1}{2}[2x-(\alpha_1+\alpha_2)]y'-(p^2x+q^2)y=0$$

$$(x-\alpha_1)(x-\alpha_2)y''+\frac{1}{2}[2x-(\alpha_1+\alpha_2)]y'-(k^2x^2-p^2x+q^2)y=0$$

x minus alpha-one times *x* minus alpha-two times second derivative of *y* plus one over two, times two times *x* minus the sum of alpha-one and alpha-two times the derivative of *y* minus p squared times *x* plus *q* squared times *y* equals zero

x minus alpha-one times *x* minus alpha-two, times second derivative of *y* plus one over two, times two times *x* minus the sum of alpha-one and alpha-two times the derivative of *y* minus in brackets *k* squared times *x* squared minus *p* squared times *x* plus *q* squared close brackets times *y* equals zero

Bernoulli differential equation

$$y'-p(x)y=q(x)y^n\ ,\ n\in R\setminus\{0,1\}$$

derivative of *y* plus function *p* of *x* times *y* equals function *q* of *x* times *y* to the *n*-th power, where $n\in R\setminus\{0,1\}$

Bessel differential equation

$$x^2y''+xy'+(\lambda^2x^2-n^2)y=0$$

x squared times second derivative of *y* plus *x* times first derivative of *y* plus product of lambda squared times *x* squared minus *n* squared and *y* equals zero

Binomial differential equation

$$(y')^m=f(x,y)$$

derivative of *y* to the *m*-th power equals function *f* of arguments *x* and *y*

Bôcher equation

$$y''+\frac{1}{2}\left[\frac{m_1}{x-\alpha_1}+\ldots+\frac{m_{n-1}}{x-\alpha_{n-1}}\right]y'+\frac{1}{4}\left[\frac{A_0+A_1x+\ldots+A_lx^l}{(x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_{n-1})^{m_{n-1}}}\right]y=0$$

second derivative of *y* plus one over two times she sum *m*-one divided by *x* minus alpha-one plus and so on plus *m-n* minus one divided by *x* minus alpha-*n* minus one times derivative of *y* plus one over four times the sum *A*-zero plus *A*-one times *x* plus and so on plus *A*-l times *x* to the l-th power all divided by *x* – alpha-one to the power of *m*-one times *x* minus alpha-two to the power of *m*-two times and so on times *x* minus alpha-*n* minus one to the power of *m-n* minus one, the fraction then multiplied by *y* all equals zero

Briot-Bouquet equation

$$x^my'=f(x,y)$$

x to the *m*-th power times derivative of *y* equal function *f* of arguments *x* and *y*

Chebyshev differential equation

$$(1-x^2)y''-xy'+\alpha^2y=0$$

one minus x squared times the second derivative of y minus x times derivative of y plus alpha squared y equals zero

Clairaut's differential equation

$$y = xy' + f(y')$$

y equals x times derivative of y plus function f of first derivative of y

Confluent hypergeometric differential equation

$$x + y'' + (c - x)y'\alpha y = 0$$

x times second derivative of y plus c minus x times first derivative of y minus alpha y equals zero

d'Alembert's equation

$$y = xf(y') + g(y')$$

y equals x times function f of the derivative of y plus function g of the derivative of y

Duffing differential equation

$$y'' + \omega_0^2 y + \beta y^3 = 0$$

second derivative of y plus omega-zero squared times y plus beta times y cubed equals zero

Eckart differential equation

$$y'' + \left[\frac{\alpha\eta}{1+\eta} + \frac{\beta\eta}{(1+\eta)^2} + \gamma\right]y = 0 \quad \text{where} \quad \eta = e^{\delta x}.$$

second derivative of y plus in square brackets alpha times eta divided betha one plus eta plus beta times eta divided by one plus eta and the whole sum squared plus gamma close square brackets times y equals zero; eta is equal to e to the power of delta times x

Emden-Fowler differential equation

$$(x^p y')' \pm x^\sigma y^n = 0$$

the derivative of product of x to the power of p times the derivative of x plus-minus x to the power of sigma times y to the power of n equals zero

Euler differential equation

$$x^2 y'' + \alpha x y' + by = S(x)$$

x squared times the second derivative of y plus alpha times x times the derivative of y plus b times y equal function S of x

9. Work in pairs. Decide who is person A and who is B. A – go to Appendix 3A, B – go to Appendix 3B. Read your text and prepare questions about the missing information in the gaps. Ask your partner questions and fill in the gaps. When you finish check your text with your partner.

Now – try reading aloud:

1. The differential equation $y' = 1/y$ has a solution in the interval $0 < c < y < d$, because all the conditions are satisfied with $h(y) = 1/y$. Its *isoclines* are lines parallel to the x-axis.

The integral curve of $\frac{\mathrm{d}x}{\mathrm{d}y} = y$ passing through the point (ξ, η) with $\eta > 0$ in the strip {$-\infty < x < +\infty$; $c < y < d$} is obtained by solving for y from $x = \xi + \int_{\eta}^{y} y\mathrm{d}y = \xi + \frac{1}{2}(y^2 - \eta^2)$ for $y > 0$.

This gives $y = \sqrt{\{\eta^2 + 2(x - \xi)\}}$ for $x > \xi - \eta^2/2$. As was to be expected from the direction filed, the integral curves are parabolas.

---

2. *a n* of *x* times the *n*-th order derivative of function *y* of *x* plus a *n*–1 of x times the *n*–1 order derivative of *y* plus … plus a 1 of *x* times the derivative of *y* plus a 0 of *x* times *y* equals function *f* of argument *x*

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

---

3. *h* is a function of argument *y* which equals the inverse of *y*

   $h(y) = 1/y$

---

4. The derivative of function x with respect to y equals y

$$\frac{\mathrm{d}x}{\mathrm{d}y} = y$$

---

5. *y* equals the square root of eta squared plus 2 times open brackets *x* minus xi close brackets – for *x* greater than xi minus eta squared divided by 2

   $y = \sqrt{\eta^2 + 2(x - \xi)}$ for $x > \xi - \eta^2/2$.

Chapter 12

☺ **"*Q*: *Why do you rarely find mathematicians spending time at the beach?***
***A*: *Because they have sine and cosine to get a tan, and don't need the sun!*"☺**

Abstract algebra. Part 1

Lead-in:

1. What is abstract algebra?

2. Listen and repeat after the recording:
module, *quasigroup*, *binary operation*, *monoid*, *lattice*, *dyadic operation*, *semigroup*, *exponentiation*, *associativity*, *juxtaposition*, *axiom.*

3. Read the text, find any other new words.

Text:

Abstract algebra is the field of mathematics concerned with the study of algebraic structures, such as groups, rings, fields, modules, vector spaces, and algebras. Structures of this sort are defined formally, starting in the nineteenth century.

Abstract algebra, in its early life at the start of the twentieth century, was more often called modern algebra. Its study was part of the drive for more intellectual rigor in mathematics. Initially, the logical assumptions in classical algebra, on which the whole of mathematics (and major parts of the natural sciences) depend, were written out, as axiomatic systems. On that basis disciplines such as group theory and ring theory took their places in pure mathematics. The term abstract algebra is now used to distinguish the aggregate of such fields from the elementary algebra ("high school algebra"), which teaches the correct rules for manipulating formulas and algebraic expressions involving real and complex numbers, and unknowns. Elementary algebra can be taken to be an introductory branch of commutative algebra.

Contemporary mathematics and mathematical physics constantly and intensively use the results of abstract algebra; for example, the theory of Lie algebras, an abstract structure only isolated towards the end of the nineteenth century by Sophus Lie. Fields such as algebraic number theory, algebraic topology and algebraic geometry apply algebraic methods in other areas. The idea of representation theory in mathematics is, roughly speaking, to take the "abstract" out of "abstract algebra", studying the concrete side of a given structure.

The term abstract algebra is sometimes used in universal algebra, a general theory of algebra, where most authors use simply the term "algebra".

History and examples

Historically, algebraic structures usually arose first in some other field of mathematics, were specified axiomatically, and were then studied in their own right in abstract algebra. Because of this, abstract algebra has numerous fruitful connections to all other branches of mathematics.
Examples of algebraic structures with a single binary operation are:
1) quasigroups,
2) monoids,
3) semigroups and, most important, groups.

More complicated examples include:
1) rings and fields,
2) modules and vector spaces,
3) algebras over fields,
4) associative algebras and Lie algebras,
5) lattices and Boolean algebras.

In universal algebra, all those definitions and facts are collected that apply to all algebraic structures alike. All the above classes of objects, together with the proper notion of homomorphism, form categories, and category theory frequently provides the formalism for translating between and comparing different algebraic structures.

An example

The systematic study of algebra has allowed mathematicians to bring under a common logical description apparently disparate (different) conceptions. For example, consider two rather distinct operations: the composition of functions, $f(g(x))$, and the multiplication of matrices, AB. These two operations are, in fact, the same. To see this, think about multiplying two square matrices (AB) by a one-column vector, **x**. This, in fact, defines a function that is equivalent to composing Ay with Bx: Ay = A(Bx) = (AB)x. Functions under composition and matrices under multiplication form sets called monoids; a monoid under an operation is associative for all its elements ($(ab)c = a(bc)$) and contains an element e such that, for any a, $ae = ea =$ a.

Adapted from Wikipedia

Follow-up exercises

4. Answer the following questions:
   a. What does abstract algebra deal with?
   b. When did abstract algebra originate?
   c. What was abstract algebra called before?
   d. Why did the study of abstract algebra develop?
   e. In what way, in classical algebra, were the logical assumptions written out initially?
   f. Why is the term abstract algebra used now?
   g. Where do algebraic structures come from historically?
   h. What are some of the examples of algebraic structures?
   i. Are the composition of functions, $f(g(x))$, and the multiplication of matrices, AB, the same or different?

j. What are monoids?

5. Group activity: listen to your teacher, remember which number you are. (Texts 1–4).

6. Find five difficult words from this lesson. Write their definitions with your partner. Join another pair; in groups of four read your definitions, ask the other pair to guess the word. Swap roles.

7. In pairs write 10 true/false sentences to the four texts. When you are ready work with another pair, show them your sentences asking them to identify whether they are true or false.

***Text* 1**

Binary operation

In mathematics, a binary operation is a calculation involving two input quantities, in other words, an operation whose arity is two. Binary operations can be accomplished using either a binary function or binary operator. Binary operations are sometimes called dyadic operations in order to avoid confusion with the binary numeral system. Examples include the familiar arithmetic operations of addition, subtraction, multiplication and division.

More precisely, a binary operation on a set S is a binary function from S and S to S, in other words a function f from the Cartesian product S × S to S. Sometimes, especially in computer science, the term is used for any binary function. That f takes values in the same set S that provides its arguments is the property of closure.

Many binary operations of interest in both algebra and formal logic are commutative or associative. Many also have identity elements and inverse elements. Typical examples of binary operations are the addition (+) and multiplication (*) of numbers and matrices as well as composition of functions on a single set.

Examples of operations that are not commutative are subtraction (–), division (/) and exponentiation (^). Binary operations are often written using infix notation such as a * b, $a + b$, or $a \bullet b$ rather than by functional notation of the form $f(a,b)$. Sometimes they are even written just by juxtaposition: ab. They can also be expressed using prefix or postfix notations. A prefix notation, Polish notation, dispenses with parentheses; it is probably more often encountered now in its postfix form, reverse Polish notation.

***Text* 2**

Semigroup

In mathematics, a semigroup is an algebraic structure consisting of a set S closed under an associative binary operation.

Juxtaposition suffices to denote the semigroup operation. That is, xy denotes the result of applying the semigroup operation to the ordered pair (x, y).

A semigroup with an identity element is a monoid. Any semigroup S may be turned into a monoid simply by adjoining an element e not in S and defining $es = s = se$ for all $s \in S \cup \{e\}$. Some require that a semigroup have an identity element, which would render semigroups identical to monoids. Moreover, not all agree that S should be nonempty. This entry assumes that a semigroup may be empty, and need not have an identity.

***Text* 3**

Monoid

In abstract algebra, a monoid is an algebraic structure with a single, associative binary operation and an identity element. In other words, it is a unital semigroup.

Definition

A monoid is a set M with binary operation $* : M \times M \to M$, obeying the following axioms:

- associativity: for all a, b, c in M, (a*b)*c = a*(b*c);
- identity element: there exists an element e in M, such that for all a in M, a*e = e*a = a.

One often sees the additional axiom

- closure: for all a, b in M, a*b is in M

though, strictly speaking, this isn't necessary as it is implied by the notion of a binary operation.

Alternatively, a monoid is a semigroup with an identity element.
A monoid satisfies all the axioms of a group with the exception of having inverses. A monoid with inverses is the same thing as a group.
A monoid whose operation is commutative is called a commutative monoid (or, less commonly, an abelian monoid).

***Text* 4**

Group (mathematics)

In mathematics, a group is a set, together with a binary operation, such as multiplication or addition, satisfying certain axioms, detailed below. For example, the set of integers is a group under the operation of addition. The branch of mathematics which studies groups is called group theory.
The historical origin of group theory goes back to the works of Évariste Galois (1830), concerning the problem of when an algebraic equation is soluble by radicals. Previous to this work, groups were mainly studied concretely, in the form of permutations; some aspects of abelian group theory were known in the theory of quadratic forms.
Many of the objects investigated in mathematics turn out to be groups. These include familiar number systems, such as the integers, the rational numbers, the real numbers, and the complex numbers under addition, as well as the non-zero rationals, reals, and complex numbers, under multiplication. Another important example is given by non-singular matrices under multiplication, and more generally, invertible functions under composition. Group theory allows for the properties of these systems and many others to be investigated in a more general setting, and its results are widely applicable. Group theory is also a rich source of theorems in its own right.

Chapter 13

☺ ***There are ten kinds of mathematicians.***
Those who can think binarily and those who can't... ☺

Abstract algebra. Part 2

Lead-in:

1. Quiz each other in pairs on 10 expressions from Chapter 11 (Abstract Algebra Part 1).

2. Repeat after the recording:
nonempty, *binary*, *closure*, *multiplicative*, *requirement*, *unmentioned*, *analogous*, *additive*, *commutativity*.

3. Read the following text, write down any new words.

Text:

Basic definitions

A **group** $(G, *)$ is a nonempty set G together with a binary operation $*: G \times G \rightarrow G$, satisfying the group axioms below. "$a * b$" represents the result of applying the operation * to the ordered pair (a, b) of elements of G. The group axioms are the following:

- associativity: for all a, b and c in G, $(a * b) * c = a * (b * c)$;
- neutral element: there is an element e in G such that for all a in G, $e * a = a * e = a$;
- inverse element: for all a in G, there is an element b in G such that $a * b = b * a = e$, where e is the neutral element from the previous axiom.

You will often also see the axiom:

- closure: for all a and b in G, $a * b$ belongs to G.

The way that the definition above is phrased, this axiom is not necessary, since binary operations are already required to satisfy closure. When determining if * is a group operation, however, it is nonetheless necessary to verify that * satisfies closure; this is part of verifying that it is in fact a binary operation.
The neutral element is usually called the identity element for a multiplicative group and the null element or zero element for an additive group.
It should be noted that there is no requirement that the group operation be commutative, that is there may exist elements such that $a * b \neq b * a$. A group G is said to be abelian (after the mathematician Niels Abel) (or commutative) if for every a, b in G, $a * b = b * a$. Groups lacking this property are called non-abelian.
The order of a group G, denoted by $|G|$ or $o(G)$, is the number of elements of the set G. A group is called finite if it has finitely many elements, that is if the set G is a finite set.

Note that we often refer to the group (G, *) as simply "G", leaving the operation * unmentioned. But to be perfectly precise, different operations on the same set define different groups.

Notation for groups

Usually the operation, whatever it really is, is thought of as an analogue of multiplication, and the group operations are therefore written multiplicatively. That is:

- we write "$a \bullet b$" or even "ab" for $a * b$ and call it the product of a and b;
- we write "1" (or "e") for the neutral element and call it the unit element;
- we write "a^{-1}" for the inverse of a and call it the reciprocal of a.

However, sometimes the group operation is thought of as analogous to addition and written additively:

- we write "$a + b$" for $a * b$ and call it the sum of a and b;
- we write "0" for the neutral element and call it the zero element;
- we write "$-a$" for the inverse of a and call it the opposite of a.

Usually, only abelian groups are written additively, although abelian groups may also be written multiplicatively. When being noncommittal, one can use the notation (with "*") and terminology that was introduced in the definition, using the notation a^{-1} for the inverse of a. If S is a subset of G and x an element of G, then, in multiplicative notation, xS is the set of all products $\{xs : s \text{ in } S\}$; similarly the notation $Sx = \{sx : s \text{ in } S\}$; and for two subsets S and T of G, we write ST for $\{st : s \text{ in } S, t \text{ in } T\}$. In additive notation, we write $x + S$, $S + x$, and $S + T$ for the respective sets.

Ring (mathematics)

In mathematics, a ring is an algebraic structure in which addition and multiplication are defined and have properties listed below. A ring is a generalization of the integers, which are one example of a ring. Other examples include the polynomials and the integers modulo n. The branch of abstract algebra which studies rings is called ring theory.

Task:

Try to read the following operations. Find the difficulties. Compare with the recording.

Formal definition

A ring is a set R equipped with two binary operations + and •, conventianally called addition and multiplication, such that:

- (R, +) is an abelian group with zero element:
  - — $(a + b) + c = a + (b + c)$,
  - — $a + b = b + a$,
  - — $0 + a = a + 0 = a$,
  - — for every a in R, there exists an element denoted $-a$, such that $a + -a = -a + a = 0$;
- (R, •) is a semigroup:
  - — $(a \bullet b) \bullet c = a \bullet (b \bullet c)$,
- Multiplication distributes over addition:
  - — $a \bullet (b + c) = (a \bullet b) + (a \bullet c)$,

— $(a + b) \bullet c = (a \bullet c) + (b \bullet c)$.

As with groups the symbol • is usually omitted and multiplication is just denoted by juxtaposition. Also the standard order of operation rules are used, so that for example, $a + bc$ is an abbreviation for $a + (b \bullet c)$.
Although ring addition is commutative, such that $a + b = b + a$, ring multiplication is not required to be commutatitve – $a \bullet b$ need not equal $b \bullet a$. Rings that also satisfy commutativity for multiplication (such as the ring of integers) are called commutative rings. Not all rings are commutative. For example $M_n(K)$, the ring of $n \times n$ matrices over a field K, is a non-commutative ring.
Rings need not have multiplicative inverses either. An element a in a ring is called a unit if it is invertible with respect to multiplication, such that there is an element b in the ring such that $a \bullet b = b \bullet a = 1$. If that is the case, then b is uniquely determined by a and we write $a^{-1} = b$. The set of all units in R forms a group under ring multiplication; this group is denoted by $U(R)$.

Field (mathematics)

A field is a commutative ring $(F, +, *)$ such that 0 does not equal 1 and all elements of F except 0 have a multiplicative inverse. (Note that 0 and 1 here stand for the identity elements for the + and * operations, respectively, which may differ from the familiar real numbers 0 and 1).

Spelled out, this means that the following hold:

- closure of F under + and *
  - — for all a, b belonging to F, both $a + b$ and $a * b$ belong to F (or more formally, + and * are binary operations on F);
- both + and * are associative
  - — for all a, b, c in F, $a + (b + c) = (a + b) + c$ and $a * (b * c) = (a * b) * c$;
- both + and * are commutative
  - — for all a, b belonging to F, $a + b = b + a$ and $a * b = b * a$;
- the operation * is distributive over the operation +
  - — for all $a, b, c,$ belonging to F, $a * (b + c) = (a * b) + (a * c)$;
- existence of an additive identity
  - — there exists an element 0 in F, such that for all a belonging to F, $a + 0 = a$;
- existence of a multiplicative identity
  - — there exists an element 1 in F different from 0, such that for all a belonging to F, $a * 1 = a$;
- existence of additive inverses
  - — for every a belonging to F, there exists an element $-a$ in F, such that $a + (-a) = 0$;
- existence of multiplicative inverses
  - — for every $a \neq 0$ belonging to F, there exists an element a^{-1} in F, such that $a * a^{-1} = 1$.

The requirement $0 \neq 1$ ensures that the set which only contains a single element is not a field. Directly from the axioms, one may show that $(F, +)$ and $(F - \{0\}, *)$ are commutative groups (abelian groups) and that therefore (see elementary group theory) the additive inverse $-a$ and the multiplicative inverse a^{-1} are uniquely determined by a. Furthermore, the multiplicative inverse of a product is equal to the product of the inverses:

$$(a * b)^{-1} = b^{-1} * a^{-1} = a^{-1} * b^{-1}$$

provided both a and b are non-zero. Other useful rules include

$$-a = (-1) * a$$

and more generally

$$-(a * b) = (-a) * b = a * (-b)$$

as well as

$$a * 0 = 0,$$

all rules familiar from elementary arithmetic.

If the requirement of commutativity of the operation * is dropped, one distinguishes the above commutative fields from non-commutative fields, usually called division rings or skew fields.

Adapted from Wikipedia

Follow-up exercises

4. Prepare 5 questions to the text. Work in groups of three, asking and answering the questions.

5. Make a list of important terms. Quiz each other in pairs, trying to define the words.

6. Rearrange the following words to form correct sentences:

a) identity neutral is The called the element usually element.
b) has many A is finite if it finitely group elements called.
c) operations on same define the different set groups Different.
d) algebraic A ring is structure in addition an and multiplication are which defined.
e) algebra The of abstract which studies is branch ring called theory rings.
f) called An element *a* a ring is a unit in if is invertible with it respect multiplication to.
g) $0 \neq 1$The requirement set ensures that the which field only contains a element single is not a.
h) product The multiplicative equal inverse of a is to the inverses product of the.

7. Fill in the missing nouns and verbs:

NOUN	**VERB**
operation	__________
__________	satisfy
__________	represent
application	__________
__________	associate
definition	__________
__________	compose
multiplication	__________

8. Find factual mistakes in each of the following sentences:

a) the neutral element is usually called the axiom for a multiplicative group, and the null element or zero element for an additive group;
b) a group is called finite if it has infinitely many elements;
c) usually the operation, whatever it really is, is thought of as an analogue of division, and the group operations are therefore written multiplicatively;
d) in mathematics, a ring is an algebraic element in which addition and multiplication are defined;
e) the branch of abstract algebra which studies fields is called ring theory;
f) a ring is a set R equipped with two binary operations + and *, called deduction and multiplication;
g) an element in a ring is called a set if it is invertible with respect to multiplication;
h) the requirement $0 \neq 1$ ensures that the set which only contains a double element is not a field.

Now – try reading aloud:

1. Ordinary addition and multiplication are operations on the sets of integers, rational numbers, real numbers, and complex numbers. Matrix multiplication is an operation on the set of all $(n \times n)$-matrices, on the set of $(n \times n)$-matrices with non-zero determinant, and on the set of $(n \times n)$-matrices whose determinant is 1.
An operation can be defined on the set of permutations of a certain fixed number of objects, by defining the product of two permutations to be the permutation obtained by performing one after another (this is a special case of the composition of mappings). For the permutations $p_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$ and $p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$ the product is $p_1 \cdot p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$, as can be seen by the following scheme, in which effect on the objects is made clear
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}.$$
The product of two permutations of a fixed number of objects is another permutation of those objects.

---

2. y equals the square root of eta squared plus 2 times, open brackets x minus xi, close brackets, for x greater than xi minus eta squared divided by 2
$$y = \sqrt{\eta 2 + 2(x - \xi)} \quad \text{for} \quad x > \xi - \eta^2/2 \,,$$

---

3. s is an element of the union of set S and the set consisting of element e
$s \in S \cup \{e\}$,

---

4. Permutation *p* 1 maps 1 to 2, 2 to 3, 3 to 1 and 4 to 4; permutation *p* 2 maps 1 to 4, 2 to 1, 3 to 3 and 4 to 2

$$p_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \quad \text{and} \quad p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix},$$

---

5. The product of permutations *p* 1 and p 2 is a permutation that maps 1 to 1, 2 to 3, 3 to 4 and 4 to 2

$$p_1 \cdot p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}.$$

Long and short scales

The long and short scales are two different numerical systems used throughout the world:

> **Short scale** is the English translation of the French term échelle courte.[1] It refers to a system of numeric names in which every new term is 1,000 times greater than the previous term: "billion" means "a thousand millions" (10^9), "trillion" means "a thousand billions" (10^{12}), and so on.
>
> **Long scale** is the English translation of the French term échelle longue. It refers to a system of numeric names in which every new term is 1,000,000 times greater than the previous term: "Billion" means "a million millions" (10^{12}), "trillion" means "a million billions" (10^{18}), and so on.

For most of the 19th and 20th centuries, the United Kingdom uniformly used the long scale, while the United States of America used the short scale, so the two systems were often referred to (and accurately at that time) as "British" and "American" usage, respectively. Today, the UK uses the short scale exclusively in official and mass media usage, although some long-scale usage still continues. Both scales were used in France and Italy at various times in history, but these countries and most other European countries now officially use long scale.

The existence of these different scales means that great care must be taken when translating large numbers between languages and using old documents in countries where the dominant scale has changed over time. Apparently identical words in different languages may actually mean substantially different values. For example, the French word "billion" (10^{12}) translates to the English word "trillion" (usually 10^{12}), not "billion" (usually 10^9). The German word "Billion" and the Dutch word "biljoen" refer to 10^{12}. Large numbers in historical documents may imply a different value than the current usage. British English documents from 1900 used long scale values, which are different from current British short scale usage, since British English has predominantly changed from long scale to short scale usage over this time period. American English books from 1900 contain the same short scale usage of billion as today.

Note that the difference between the two scales grows as numbers get larger. The long-scale billion is a thousand times larger than the short-scale billion, but the long-scale trillion is a million times larger than the short-scale trillion, and so on.

Comparison

Value	Short Scale	Long Scale
10^0 = 1	one	one
10^3 = 1000	thousand	thousand
10^6 = 1000,000	million	million
10^9 = 1000,000,000	billion	thousand million (or milliard)
10^{12} = 1000,000,000,000	trillion	billion
10^{15} = 1000,000,000,000,000	quadrillion	thousand billion
10^{18} = 1000,000,000,000,000,000	quintillion	trillion

Since bi refers to 2 and tri refers to 3, the logic in the names is:

- short scale: Billion is thousand times thousand 2 = 10^9. Trillion is thousand times thousand 3 = 10^{12}.

To get from one named order of magnitude to the next, multiply by a thousand.

- long scale: Million is million^1 = 10^6. Billion is million^2 = 10^{12}. Trillion is million^3 = 10^{18}.

To get from one named order of magnitude to the next, multiply by a million.
The old word “milliard”, also found in many other languages, can be used for 10^9 but is unknown in American English and not used in British English – however, “Yard”, which derives from “milliard”, is used on financial markets, as unlike ‘billion’, it is unambiguous.

Short scale countries

English language speaking countries

Most English language countries currently use the short scale. For example:

- United States,
- English-speaking Canada,
- United Kingdom – albeit with some residual usage of the long-scale,
- Republic of Ireland,
- Australia,
- New Zealand.

Other languages and countries:

Brazil, which despite speaking Portuguese, uses 10^9 = bilhão, 10^{12} = trilhão, etc.

Long scale countries

Many languages in the world continue to use the long scale, with many using milliard to mean 10^9, and/or billion to mean 10^{12}. This is particularly common in languages derived from Scandinavia and/or Continental Europe. Examples include:

- 10^9 Milliard: French, Danish and Norwegian milliard, German Milliarde, Dutch miljard, Hungarian milliárd (although “trillion” is perceived to be 1000 billions, and so on, so not really a long scale country), Hebrew milliard, Spanish millardo (but more frequently “mil millones”), Italian miliardo, Polish miliard, Finnish miljardi, Lithuanian milijardas, Czech and Slovak miliarda, Romanian miliard, Slovenian, Croatian, Serbian milijarda, Icelandic milljarður, Turkish milyar.

- 10^{12} Billion: French, Danish and Norwegian billion, German Billion, Dutch biljoen, Hungarian billió, Spanish billón, Polish and Serbian bilion, Finnish biljoona, Croatian bilijun, European Portuguese bilião, Slovenian bilijon, Icelandic billjón.

Both long and short scales countries

Short scale use with long scale million

Some countries use the short scale in which the short scale billion is replaced with million, as in the long scale. In this usage, the numbering would be 10^3 = one thousand, 10^6 = one thousand thousand, 10^9 = one million, 10^{12} = one trillion.

— Bulgaria (милиард)
— Iran (milliard)
— Latvia (miljards)
— Russia (миллиард)
— Turkey (milyar)

Short scale and long scale

- Puerto Rico, a Spanish-speaking US Commonwealth country, generally uses short scale (10^9 = billón, 10^{12} = trillón) in economic and technical matters, but the long scale is used in publications intended for a Latin American audience outside Puerto Rico.

Neither short nor long scale countries

The following countries have their own numbering systems and use neither short nor long scales:

- China – see Chinese large numbers. which features symbols for all the myriads up to 10^{44}.
- India – see Indian numbering system which is commonly used. For Indian English speakers see below.
- Japan – see Japanese numerals: powers of 10 which uses myriads as in Chinese.
- Korea – see Korean numerals which uses a traditional myriad system for the larger numbers, with special words and symbols up to 10^{48}.
- Greece – see Greek numerals which uses a traditional myriad system, with 10^9 = disekatommyrio (“bi-hundred-myriad”), 10^{12} = trisekatommyrio, (“tri-hundred-myriad”), etc.

Adapted from Wikipedia, the free encyclopedia

Basic mathematical symbols

Symbol	Name Should be read as Category	Explanation	Examples
=	equality is equal to; equals Everywhere	$x = y$ means x and y represent the same thing or value.	$1 + 1 = 2$
≠ < >	inequality is not equal to; does not equal Everywhere	$x \neq y$ means that x and y do not represent the same thing or value.	$1 \neq 2$
< > ≪ ≫	strict inequality is less than, is greater than, is much less than, is much greater than order theory	$x < y$ means x is less than y. $x > y$ means x is greater than y. $x \ll y$ means x is much less than y. $x \gg y$ means x is much greater than y.	$3 < 4$ $5 > 4$. $0.003 \ll 1{,}000{,}000$
≤ ≥	inequality is less than or equal to, is greater than or equal to order theory	$x \leq y$ means x is less than or equal to y. $x \geq y$ means x is greater than or equal to y.	$3 \leq 4$ and $5 \leq 5$ $5 \geq 4$ and $5 \geq 5$
∝	proportionality is proportional to Everywhere	$y \propto x$ means that $y = kx$ for some constant k.	if $y = 2x$, then $y \propto x$
+	addition Plus arithmetic disjoint union the disjoint union of … and … set theory	4 + 6 means the sum of 4 and 6. $A_1 + A_2$ means the disjoint union of sets A_1 and A_2.	$2 + 7 = 9$ $A_1=\{1,2,3,4\} \wedge A_2=\{2,4,5,7\} \Rightarrow A_1 + A_2 = \{(1,1), (2,1), (3,1), (4,1), (2,2), (4,2), (5,2), (7,2)\}$
−	subtraction Minus arithmetic negative sign Negative; minus arithmetic set-theoretic complement minus; without set theory	9 − 4 means the subtraction of 4 from 9. −3 means the negative of the number 3. $A - B$ means the set that contains all the elements of A that are not in B.	$8 - 3 = 5$ $-(-5) = 5$ $\{1,2,4\} - \{1,3,4\} = \{2\}$

×	multiplication Times arithmetic Cartesian product the Cartesian product of … and …; the direct product of … and … set theory cross product Cross vector algebra	3 × 4 means the multiplication of 3 by 4. $X \times Y$ means the set of all ordered pairs with the first element of each pair selected from X and the second element selected from Y. $u \times v$ means the cross product of vectors u and v	7 × 8 = 56 {1,2} × {3,4} = {(1,3), (1,4), (2,3), (2,4)} (1,2,5) × (3,4,−1) = (−22, 16, − 2)
÷ /	division divided by arithmetic	6 ÷ 3 or 6/3 means the division of 6 by 3.	2 ÷ 4 = .5 12/4 = 3
$\sqrt{\ }$	square root the principal square root of; square root real numbers complex square root the complex square root of; square root complex numbers	$\sqrt{x}$ means the positive number whose square is x. if $z = r\exp(i\varphi)$ is represented in polar coordinates with $-\pi < \varphi \le \pi$, then $\sqrt{z} = \sqrt{r}\exp(i\varphi/2)$.	$\sqrt{4} = 2$ $\sqrt{(-1)} = i$
| |	absolute value Modulus of numbers	$\|x\|$ means the distance in the real line (or the complex plane) between x and zero.	$\|3\| = 3$, $\|-5\| = \|5\|$ $\|i\| = 1$, $\|3+4i\| = 5$
!	factorial Factorial combinatorics	n! is the product 1 × 2 × ... × n.	4! = 1 × 2 × 3 × 4 = 24
~	probability distribution has distribution statistics	$X \sim D$, means the random variable X has the probability distribution D.	$X \sim N(0,1)$, the standard normal distribution
⇒ → ⊃	material implication implies; if .. then propositional logic	$A \Rightarrow B$ means if A is true then B is also true; if A is false then nothing is said about B. → may mean the same as ⇒, or it may have the meaning for functions given below. ⊃ may mean the same as ⇒, or it may have the meaning for superset given below.	$x = 2 \Rightarrow x^2 = 4$ is true, but $x^2 = 4 \Rightarrow x = 2$ is in general false (since x could be −2).
⇔ ↔	material equivalence if and only if; iff propositional logic	$A \Leftrightarrow B$ means A is true if B is true and A is false if B is false.	$x + 5 = y + 2 \Leftrightarrow x + 3 = y$

¬ ~	logical negation Not propositional logic	The statement ¬*A* is true if and only if *A* is false. A slash placed through another operator is the same as "¬" placed in front.	$\neg(\neg A) \Leftrightarrow A$ $x \neq y \Leftrightarrow \neg(x = y)$
∧	logical conjunction or meet in a lattice And propositional logic, lattice theory	The statement $A \wedge B$ is true if *A* and *B* are both true; else it is false.	$n < 4 \wedge n > 2 \Leftrightarrow n = 3$ when n is a natural number.
∨	logical disjunction or join in a lattice Or propositional logic, lattice theory	The statement $A \vee B$ is true if *A* or *B* (or both) are true; if both are false, the statement is false.	$n \geq 4 \vee n \leq 2 \Leftrightarrow n \neq 3$ when n is a natural number.
⊕ ⊻	exclusive or Xor propositional logic, Boolean algebra	The statement $A \oplus B$ is true when either *A* or *B*, but not both, are true. $A \veebar B$ means the same.	$(\neg A) \oplus A$ is always true, $A \oplus A$ is always false.
∀	universal quantification for all; for any; for each predicate logic	$\forall\, x$: *P*(*x*) means *P*(*x*) is true for all x.	$\forall\, n \in N$: $n^2 \geq n$.
∃	existential quantification there exists predicate logic	$\exists\, x$: *P*(*x*) means there is at least one x such that *P*(*x*) is true.	$\exists\, n \in N$: *n* is even.
∃!	uniqueness quantification there exists exactly one predicate logic	$\exists!\, x$: *P*(*x*) means there is exactly one x such that *P*(*x*) is true.	$\exists!\, n \in N$: $n + 5 = 2n$.
:= ≡ :⇔	definition is defined as everywhere	$x := y$ or $x \equiv y$ means *x* is defined to be another name for *y* (but note that ≡ can also mean other things, such as congruence). $P :\Leftrightarrow Q$ means *P* is defined to be logically equivalent to *Q*.	$\cosh x := (1/2)(\exp x + \exp(-x))$ *A* XOR *B* :⇔ $(A \vee B) \wedge \neg(A \wedge B)$
{ , }	set brackets the set of ... set theory	$\{a, b, c\}$ means the set consisting of *a*, *b*, and *c*.	N = {0,1,2,...}
{ : } { | }	set builder notation the set of ... such that ... set theory	$\{x : P(x)\}$ means the set of all *x* for which *P*(*x*) is true. $\{x \mid P(x)\}$ is the same as $\{x : P(x)\}$.	$\{n \in N : n^2 < 20\} = \{0,1,2,3,4\}$

∅ {}	empty set the empty set set theory	∅ means the set with no elements. {} means the same.	$\{n \in N : 1 < n^2 < 4\} = \varnothing$
∈ ∉	set membership is an element of; is not an element of, belongs to everywhere, set theory	$a \in S$ means a is an element of the set S; $a \notin S$ means a is not an element of S.	$(1/2)^{-1} \in N$ $2^{-1} \notin N$
⊆ ⊂	subset is a subset of set theory	(subset) $A \subseteq B$ means every element of A is also element of B. (proper subset) $A \subset B$ means $A \subseteq B$ but $A \neq B$.	$A \cap B \subseteq A$; $Q \subset R$
⊇ ⊃	superset is a superset of set theory	$A \supseteq B$ means every element of B is also element of A. $A \supset B$ means $A \supseteq B$ but $A \neq B$.	$A \cup B \supseteq B$; $R \supset Q$
∪	set-theoretic union the union of ... and ...; union set theory	(exclusive) $A \cup B$ means the set that contains all the elements from A, or all the elements from B, but not both. "A or B, but not both". (inclusive) $A \cup B$ means the set that contains all the elements from A, or all the elements from B, or all the elements from both A and B. "A or B or both".	$A \subseteq B \Leftrightarrow A \cup B = B$ (inclusive)
∩	set-theoretic intersection intersected with; intersect set theory	$A \cap B$ means the set that contains all those elements that A and B have in common.	$\{x \in \mathrm{R} : x^2 = 1\} \cap \mathrm{N} = \{1\}$
\	set-theoretic complement minus; without set theory	$A \setminus B$ means the set that contains all those elements of A that are not in B.	$\{1,2,3,4\} \setminus \{3,4,5,6\} = \{1,2\}$
( )	function application Of set theory precedence grouping everywhere	$f(x)$ means the value of the function f at the element x. Perform the operations inside the parentheses first.	If $f(x) := x^2$, then $f(3) = 3^2 = 9$. $(8/4)/2 = 2/2 = 1$, but $8/(4/2) = 8/2 = 4$.
$f{:}X \to Y$	function arrow from ... to set theory	$f: X \to Y$ means the function f maps the set X into the set Y.	Let $f: Z \to N$ be defined by $f(x) = x^2$.

∘	function composition composed with set theory	$f \circ g$ is the function, such that $(f \circ g)(x) = f(g(x))$.	if $f(x) = 2x$, and $g(x) = x + 3$, then $(f \circ g)(x) = 2(x+3)$.
N ℕ	natural numbers *N* numbers	*N* means {0,1,2,3,...}, but see the article on natural numbers for a different convention.	$\{\lvert a\rvert : a \in Z\} = N$
Z ℤ	integers *Z* numbers	*Z* means {...,−3,−2,−1,0,1,2,3,...}.	$\{a : \lvert a\rvert \in N\} = Z$
Q ℚ	rational numbers *Q* numbers	*Q* means $\{p/q : p,q \in Z, q \neq 0\}$.	$3.14 \in Q$ $\pi \notin Q$
R ℝ	irrational numbers *R* numbers	*R* means the set of real numbers.	$\pi \in R$ $\sqrt{(-1)} \notin R$
C ℂ	complex numbers *C* numbers	*C* means $\{a + bi : a,b \in R\}$.	$i = \sqrt{(-1)} \in C$
∞	infinity infinity numbers	∞ is an element of the extended number line that is greater than all real numbers; it often occurs in limits.	$\lim_{x\to 0} 1/\lvert x\rvert = \infty$
π	pi Pi Euclidean geometry	π is the ratio of a circle's circumference to its diameter. Its value is 3.1415....	$A = \pi r^2$ is the area of a circle with radius *r*
‖ ‖	norm norm of; length of linear algebra	$\lVert \mathbf{x} \rVert$ is the norm of the element $\mathbf{x}$ of a normed vector space.	$\lVert \mathbf{x} + \mathbf{y} \rVert \leq \lVert \mathbf{x} \rVert + \lVert \mathbf{y} \rVert$
∑	summation sum over ... from ... to ... of arithmetic	$\sum^{n}_{k=1} a_k$ means $a_1 + a_2 + ... + a_n$.	$\sum^{4}_{k=1} k^2 = 1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30$
∏	product product over ... from ... to ... of arithmetic Cartesian product the Cartesian product of; the direct product of set theory	$\prod^{n}_{k=1} a_k$ means $a_1 a_2 \cdots a_n$. $\prod^{n}_{i=0} Y_i$ means the set of all $(n + 1)$-tuples $(y_0,...,y_n)$.	$\prod^{4}_{k=1} (k + 2) = (1 + 2)(2 + 2)(3 + 2)(4 + 2) = 3 \times 4 \times 5 \times 6 = 360$ $\prod^{3}_{n=1} R = R^3$
′	derivative … prime; derivative of … calculus	$f'(x)$ is the derivative of the function f at the point *x*, i.*e*., the slope of the tangent to *f* at *x*.	If $f(x) = x^2$, then $f'(x) = 2x$

$\int$	indefinite integral or antiderivative indefinite integral of; the antiderivative of … calculus definite integral integral from ... to ... of ... with respect to dx calculus	$\int f(x)\,dx$ means a function whose derivative is f. $\int_a^b f(x)\,dx$ means the signed area between the x-axis and the graph of the function f between $x = a$ and $x = b$.	$\int x^2\,dx = x^3/3 + C$ $\int_0^b x^2\ dx = b^3/3$;
∇	gradient del, nabla, gradient of calculus	$\nabla f(x_1, \ldots, x_n)$ is the vector of partial derivatives $(\partial f / \partial x_1, \ldots, \partial f / \partial x_n)$.	If $f(x,y,z) = 3xy + z^2$, then $\nabla f = (3y, 3x, 2z)$
∂	partial derivative partial derivative of calculus boundary boundary of topology	With $f(x_1, \ldots, x_n)$, $\partial f / \partial x_i$ is the derivative of f with respect to x_i, with all other variables kept constant. ∂M means the boundary of M	If $f(x,y) = x^2y$, then $\partial f / \partial x = 2xy$ $\partial\{x : \Vert x \Vert \leq 2\} = \{x : \Vert x \Vert = 2\}$
$\perp$	perpendicular is perpendicular to geometry bottom element the bottom element lattice theory	$x \perp y$ means x is perpendicular to y; or more generally x is orthogonal to y. $x = \perp$ means x is the smallest element.	If $l \perp m$ and $m \perp n$ then $l \parallel n$. $\forall x : x \wedge \perp = \perp$
$\models$	entailment entails model theory	$A \models B$ means the sentence A entails the sentence B, that is every model in which A is true, B is also true.	$A \models A \vee \neg A$
$\vdash$	inference infers or is derived from propositional logic, predicate logic	$x \vdash y$ means y is derived from x.	$A \rightarrow B \vdash \neg B \rightarrow \neg A$
$\triangleleft$	normal subgroup is a normal subgroup of group theory	$N \triangleleft G$ means that N is a normal subgroup of group G.	$Z(G) \triangleleft G$
/	quotient group mod group theory	G/H means the quotient of group G modulo its subgroup H.	$\{0, a, 2a, b, b+a, b+2a\}$ / $\{0, b\} = \{\{0, b\}, \{a, b+a\}, \{2a, b+2a\}\}$
$\approx$	isomorphism is isomorphic to group theory approximately equal is approximately equal to everywhere	$G \approx H$ means that group G is isomorphic to group H $x \approx y$ means x is approximately equal to y	$Q / \{1, -1\} \approx V$, where Q is the quaternion group and V is the Klein four-group. $\pi \approx 3.14159$
$\otimes$	tensor product tensor product of linear algebra	$V \otimes U$ means the tensor product of V and U.	$\{1, 2, 3, 4\} \otimes \{1,1,1\} = \{\{1, 2, 3, 4\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4\}\}$

Appendices

Appendix 1A

Calculus was created in large part by 1) ...., although some of the ideas were already used by Fermat and even Archimedes.

Calculus is divided into two parts, which are closely related. One part is called 2) "..............." and the other part is called "integral calculus".

Integral calculus is concerned with area and volume. How do you determine the area of a circle or the volume of a sphere? Another way of putting it is: how much paint do you need to color in a circle? How much water do you need to fill up a ball? Integral calculus explains

The basic idea of integral calculus is this: the simplest shape whose area we can compute is the rectangle. The area is the length of the rectangle multiplied by its width. For instance, a "square mile" is a piece of land with as much area as a square plot of land with sides measuring 3) .... To compute the area of a more complicated region, we chop up the region into lots and lots of 4) .... When we do this, we will not be able to succeed completely because there will always be pieces with curved sides, generally. But the key idea is that the sum of the areas of the rectangular pieces will be a very close approximation of the actual area, and the more pieces we cut, the closer our approximation will be.

Differential calculus answers the following question: imagine you go 5) .... Suppose you know your 6) .... at all times. In other words, at 10 a.m. you're in the garage, at 10 a.m. and 5 seconds you're just outside the garage, at 10 a.m. and 10 seconds you're on the road just in front of your house...and so on... At the end of your trip, you realize that at every moment during your trip, your 7) ………………….. showed the speed of your car. Just from the knowledge of your position at all times, can you reconstruct what your speedometer showed at any time? The answer is, yes, you can, and differential calculus provides a method for doing this.

Adapted from: *www.mathforum.org*

Appendix 2A

Geometry Version A

Ask your partner questions to find out the words in the gaps.

Geometry is the study of 1) .... in a space of a given number of dimensions and of a given type. The most common types of geometry are 2) .... geometry (dealing with objects like the point, line, circle, triangle, and polygon), solid geometry (dealing with objects like the line, sphere, and polyhedron), and spherical geometry (dealing with objects like the spherical triangle and spherical polygon). Geometry was part of the quadrivium taught in medieval universities.

A mathematical pun notes that without geometry, life is pointless. An old children's joke asks, "What does an 3) .... say when it grows up?" and answers, "Geometry" ("gee, I'm a tree").

Historically, the study of geometry proceeds from a small number of accepted truths (axioms or postulates), then builds up 4) .... using a systematic and rigorous step-by-step proof. However, there is much more to geometry than this relatively 5) .... textbook approach, as evidenced by some of the beautiful and unexpected results of projective geometry (not to mention Schubert's powerful but questionable 6) .... geometry).

The late mathematician E. T. Bell has described geometry as follows (Coxeter and Greitzer 1967, p. 1): "With a literature much vaster than those of algebra and arithmetic combined, and at least as extensive as that of analysis, 7) .... is a richer treasure house of more interesting and half-forgotten things, which a hurried generation has no leisure to enjoy, than any other division of mathematics." While the literature of algebra, arithmetic, and analysis has grown extensively since 8) ...., the remainder of his commentary holds even more so today.

Adapted from: *www.mathworld.wolfram.com*

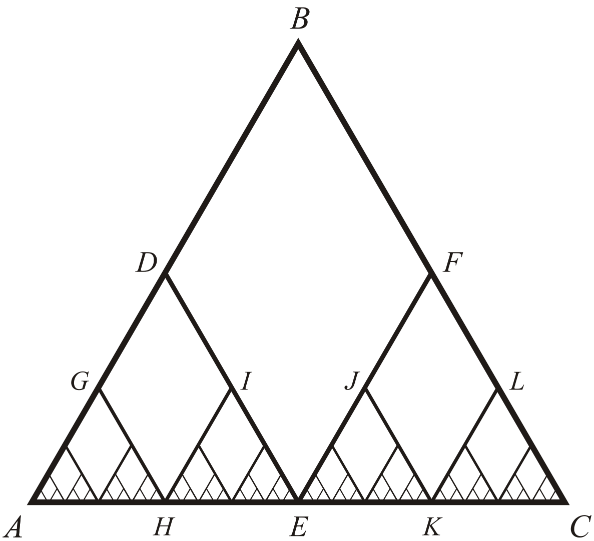
Appendix 3A

The Limit Paradox version A

Ask your partner questions to find out the words in the gaps

The Limit Paradox

There is an interesting "thought experiment" that often puzzles 1) .... when they first learn about limits in calculus. This is known as the Limit Paradox, and is sometimes presented in the form of an equilateral triangle as shown below:



The 2) .... *ABC* is twice as long as *AC*. Similarly the path *ADEFC* is also twice as long as AC, as is the path *AGHIEJKLC*, and so on. Breaking down the 3) .... path into smaller and smaller jags, the deviation of the jagged path from the straight line *AC* goes to zero, so, in a sense, the line *AC* is the 4) .... of the sequence of jagged paths. This might seem to suggest that the length of *AC* is twice the length of 5) ....!

Paradoxes like this were discussed extensively (and very seriously) during the early history of 6) .... Another example – one that may help to illustrate the fallacy of these paradoxes – is to consider the sequence of 7) ....0.9, 0.99, 0.999, etc. Clearly none of these numbers is an integer. However, these numbers approach ever more closely 8) ...., so are we justified in concluding that the number 1.0 is not an integer? No. Similarly, we could note that the average 9) .... of the non-zero decimal digits of 0.9, 0.99, etc. is 9, so we might think the average size of the non-zero digits of 1.0 must also be 10) ...., but of course it isn't.

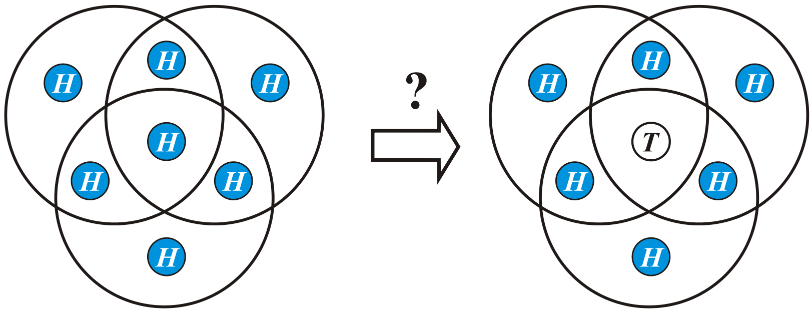
Adapted from: www.mathpages.com

Appendix 4A

Version A
Ask your partner questions to find out the word in the gaps

The Coin in Three Fountains

Draw three intersecting circles and place a coin, heads up, in each of the seven enclosed regions. Two operations are allowed: a FLIP turns over all four of the coins within a single circle, whereas a CLEAR turns all four coins in a single circle heads up. Is it possible to reach a condition in which the coin in the central region (contained within all three of the circles) is tails while all the rest are heads?



The answer is no. In fact, if, at any stage, all the coins in at least one circle are 1) ...., it's impossible to proceed to the desired final configuration by means of FLIPS and CLEARS. This not only 2) .... a wide range of initial conditions, it also implies that for 3) .... initial configurations no sequence of operations containing a CLEAR can yield the desired 4) .... configuration.

PROOF: Assume that at some stage (possibly the 5) .... configuration) all the coins in at least one circle are heads. Therefore, any sequence of FLIPS and CLEARS can be assumed to contain at least one CLEAR. Let A denote the 6) .... that is the subject of the last CLEAR in the sequence, and let B and C denote the other two circles. Every 7) .... following the last CLEAR is a FLIP of either A, B, or C. Let nA, nB, and nC denote the 8) .... of these FLIPS respectively. Since the coin in the triple-region was turned heads up by the last CLEAR, the sum (*n*A + *n*B + *n*C) must be 9) .... for the triple-region to end up tails. The double-regions of AB and AC were also turned heads up by the last CLEAR, so the sums (*n*A + *n*B) and (*n*A + *n*C) must both be even to leave these 10) .... heads up. It follows that *n*A, *n*B, and *n*C must each be odd. However, the oddness of *n*A implies that the single-region of A, which was turned heads up by the last CLEAR, ends up 11) .... Thus, the desired final 12) .... cannot be reached.

Adapted from: *www.mathpages.com*

Appendix 1B

Calculus was created in large part by Newton and Leibniz, although some of the ideas were already used by 1) ....

Calculus is divided into two parts, which are closely related. One part is called “differential calculus” and the other part is called 2) “...............”.

Integral calculus is concerned with 3) .... How do you determine the area of a circle or the volume of a sphere? Another way of putting it is: how much paint do you need to color in a circle? How much water do you need to fill up a ball? Integral calculus explains one way of computing such things.

The basic idea of integral calculus is this: the simplest shape whose area we can compute is the rectangle. The area is the length of the rectangle multiplied by its width. For instance, a 4) “...............” is a piece of land with as much area as a square plot of land with sides measuring one mile each. To compute the area of a more complicated region, we chop up the region into lots and lots of little rectangles. When we do this, we will not be able to succeed completely because there will always be pieces with 5) ...., generally. But the key idea is that the 6) .... of the areas of the rectangular pieces will be a very close approximation of the actual area, and the more pieces we cut, the closer our approximation will be.

Differential calculus answers the following question: imagine you go on a car ride. Suppose you know your position at all times. In other words, at 10 a.m. you’re in the garage, at 10 a.m. and 5 seconds you're just outside the garage, at 10 a.m. and 10 seconds you're on the road just in front of your house ... and so on ... . At the end of your trip, you realize that at every moment during your trip, your speedometer showed the speed of your car. Just from the knowledge of 7) ...., can you reconstruct what your speedometer showed at any time? The answer is, yes, you can, and differential calculus provides a method for doing this.

Adapted from: *www.mathforum.org*

Appendix 2B

Geometry version B

Ask your partner questions to find out the words in the gaps.

Geometry is the study of figures in a space of a given number of dimensions and of a given type. The most common types of geometry are plane geometry (dealing with objects like the point, line, circle, triangle, and polygon), 1) .... geometry (dealing with objects like the line, sphere, and polyhedron), and spherical geometry 2) .... Geometry was part of the quadrivium taught in 3) .... universities.

A mathematical pun notes that without geometry, life is pointless. An old children's joke asks, "What does an acorn say when it grows up?" and answers, "Geometry" ("gee, I'm a tree").

Historically, the study of geometry proceeds from a small number of 4) .... (axioms or postulates), then builds up true statements using a systematic and rigorous step-by-step 5) .... However, there is much more to geometry than this relatively dry textbook approach, as evidenced by some of the beautiful and unexpected results of 6) .... geometry (not to mention Schubert's powerful but questionable enumerative geometry).

The late mathematician E.T. Bell has described geometry as follows (Coxeter and Greitzer 1967, p. 1): "With a literature much vaster than those of algebra and arithmetic combined, and at least as 7) .... as that of analysis, geometry is a richer treasure house of more interesting and half-forgotten things, which a hurried generation has no leisure to enjoy, than any other division of mathematics." While the literature of algebra, arithmetic, and analysis 8) .... extensively since Bell's day, the remainder of his commentary holds even more so today.

Adapted from: *www.mathworld.wolfram.com*

Appendix 3B

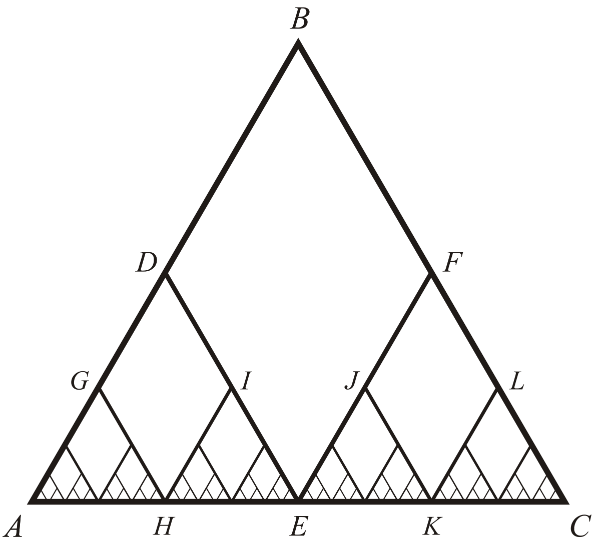
The Limit Paradox version B

Ask your partner questions to find out the words in the gaps

The Limit Paradox

There is an interesting "thought experiment" that often puzzles students when they first learn about limits in 1) .... This is known as the Limit Paradox, and is sometimes presented in the form of an 2) .... triangle as shown below:

The path *ABC* is twice as long as *AC*. Similarly the path 3) .... is also twice as long as *AC*, as is the path *AGHIEJKLC*, and so on. Breaking down the jagged path into smaller and smaller jags, the deviation of the jagged path from the straight line *AC* goes to 4) ...., so, in a sense, the line *AC* is the "limit" of the sequence of jagged paths. This might seem to suggest that the 5) .... of *AC* is twice the length of itself!



Paradoxes like this were discussed 6) .... (and very seriously) during the early history of calculus. Another example – one that may help to illustrate 7) .... of these paradoxes – is to consider the sequence of numbers 0.9, 0.99, 0.999, etc. Clearly none of these numbers is 8) .... However, these numbers approach ever more closely the number 1.0, so are we justified in concluding that the number 1.0 is not an integer? No. Similarly, we could 9) .... that the average size of the non-zero decimal digits of 0.9, 0.99, etc. is 9, so we might think the average size of the non-zero 10) .... of 1.0 must also be 9, but of course it isn't.

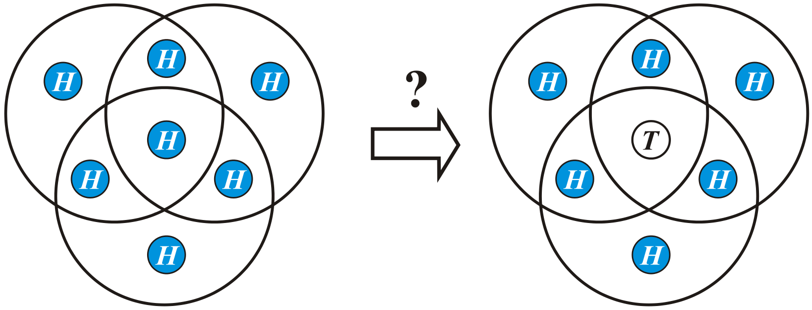
Adapted from: *www.mathpages.com*

Appendix 4B

Version B
Ask your partner questions to find out the word in the gaps.

The Coin in Three Fountains

Draw three intersecting circles and place a, 1) .... heads up, in each of the seven enclosed 2) .... Two operations are allowed: a FLIP 3) .... all four of the coins within a single circle, whereas a CLEAR turns all four coins in a single circle heads up. Is it possible to reach a 4) .... in which the coin in the central region (contained within all three of the circles) is tails while all the rest are heads?



The answer is 5) .... In fact, if, at any stage, all the coins in at least one circle are heads, it's impossible to 6) .... to the desired final configuration by means of FLIPS and CLEARS. This not only covers a wide range of 7) .... conditions, it also implies that for arbitrary initial configurations no sequence of operations containing a CLEAR can yield the desired final configuration.

PROOF: Assume that at some stage (possibly the initial configuration) all the coins in at least one circle are 8) .... Therefore, any sequence of FLIPS and CLEARS can be assumed to contain at least one CLEAR. Let A denote the circle that is the 9) .... of the last CLEAR in the sequence, and let B and C denote the other two 10) .... Every operation following the last CLEAR is a FLIP of either A, B, or C. Let *n*A, *n*B, and *n*C denote the number of these FLIPS respectively. Since the coin in the 11) .... was turned heads up by the last CLEAR, the 12) .... (*n*A + *n*B + *n*C) must be odd for the triple-region to end up tails. The double-regions of AB and AC were also turned heads up by the last CLEAR, so the sums (*n*A + *n*B) and (*n*A + *n*C) must both be even to leave these regions heads up. It follows that *n*A, *n*B, and *n*C must each be odd. However, the oddness of *n*A implies that the single-region of A, which was turned heads up by the last CLEAR, ends up tails. Thus, the desired final configuration cannot be reached.

Adapted from: *www.mathpages.com*

Appendix 5

Reading Math Operations

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1. The second(-order) derivative of function *y* of argument *x* equals derivative of function *y* of argument *x* plus a constant
   $$\frac{d^2}{dx^2}y = \frac{d}{dy}y + c.$$

---

2. The scalar product of vectors with the following coordinates 1, 2, 3 and 1, 3, 2, [1,2,3][1,3,2].

---

3. 5 minus 9 equals 5 plus minus 9, which equals minus 4
   $5 - 9 = 5 + (-9) = -4$.

---

4. 20 over 5 equals 4, minus 20 over 5 equals minus 4
   $$\frac{20}{5} = 4\ ,\ \frac{-20}{5} = -4\ .$$

---

5. 14 divided by 2 equals 7, 14 divided by minus 2 equals minus 7
   $14 \div 2 = 7\ ,\ 14 \div (-2) = -7$.

---

6. Minus 1 times minus 2 times minus 1
   $(-1)(-2)(-1)$.

---

7. Minus 3 times, open brackets, *x* plus 4, close brackets
   $-3(x + 4)$.

---

8. Minus 3 times, open brackets, *x* plus 4, close brackets equals minus 3 *x* minus 3 times 4, which equals minus 3 *x* minus 12
   $-3(x + 4) = -3x - 3\cdot 4 = -3x - 12$.

---

9. Minus 3 all squared
   $(-3)^2$.

---

10. Minus 3 all squared equals minus 3 times minus 3, which equals (positive/plus)3 times (positive/plus)3, which equals 9
    $(-3)^2 = (-3)(-3) = 3\cdot 3 = 9$.

---

11. Minus 3 all cubed
$(-3)^3$.

---

12. The square root of 16 equals 4
$\sqrt{16} = 4$.

---

13. The cube root of minus 8 equals minus 2
$\sqrt[3]{-8} = -2$.

---

14. 25 times open brackets, 1 over *a*, close brackets, to the power of minus *n* times, open brackets, 2 *n*, close brackets, to the power of 0 times 5 to (the power of) minus 3 times, open brackets, *a* over *x* close brackets, to the power of minus *n* equals 5 to the power of 2 minus 3 times a to (the power of) *n* times a to (the power of) minus *n* times *x* to (the power of) minus minus *n* equals 5 to (the power of) minus 1 times a to (the power of) 0 times *x* to (the power of) *n* equals *x* to (the power of) *n* over 5.
$$25 \cdot \left(\frac{1}{a}\right)^{-n} \cdot (2n)^0 \cdot 5^{-3} \cdot \left(\frac{a}{x}\right)^{-n} = 5^{2-3} \cdot a^n \cdot a^{-n} \cdot x^{-(-n)} = 5^{-1} \cdot a^0 \cdot x^n = \frac{x^n}{5}.$$

---

15. 27 *a* to (the power of) 4 times *b* to (the power of) 4 times 56 *a* squared times *b* to (the power of) minus 3 times 42 *a* to (the power of) minus 2 times *b* cubed equals 3 cubed times *a* to (the power of) 4 times *b* to (the power of) 4
Equals open brackets 2 squared times 3 squared times 7 *a* squared times *b* squared close brackets, squared.
$$27a^4b^4 \cdot 56a^2b^{-3} \cdot 42a^{-2}b^3$$
$$= 3^3 \cdot a^4b^4 \cdot 7 \cdot 2^3 \cdot a^2b^{-3} \cdot 7 \cdot 2 \cdot 3 \cdot a^{-2}b^3 = 2^4 \cdot 3^4 \cdot 7^2 a^4b^4 = (2^2 \cdot 3^2 \cdot 7a^2b^2)^2.$$

---

16. 2 times 10 to the power of) minus 3 times open brackets 3 times 10 to (the power of) 10 close brackets, squared over 3 point 6 times 10 to (the power of) 13, kilo watt hours equals ….
$$\frac{2 \cdot 10^{-3} \cdot (3 \cdot 10^{10})^2}{3.6 \cdot 10^{13}} \text{ kWh} = \frac{2 \cdot 9 \cdot 10^{-3} \cdot 10^{20}}{3.6 \cdot 10^{13}} \text{ kWh} = 5 \cdot 10^4 \text{ kWh}.$$

---

17. The sum of vectors **x** and **y** plus vector **z** equals vector **x** plus the sum of vectors **y** and **z**
(**x** + **y**) + **z** = **x** + (**y** + **z**).

---

18. Vector **x** multiplied by scalar *b* and then scalar *a* equals vector **x** multiplied by the product of scalars *a* and *b*
a(*b***x**) = (*ab*)**x**.

---

19. The sum of vectors **x** and **y** multiplied by scalar *a* is equal to vector **x** multiplied by *a* plus vector **y** multiplied by *a*
$a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$.

---

20. *y* equals *x* divided by 3 plus 1
$y = x/3 + 1$.

---

21. y equals *x* squared
$y = x^2$.

---

22. *y* equals the inverse of *x*
$y = 1/x$.

---

23. *m* equals *y* sub two minus *y* 1 divided by *x* 2 minus *x* 1
$m = (y_2 - y_1) / (x_2 - x_1)$.

---

24. *a* times *x* squared plus *b* times *x* times *y* plus *c* times *y* squared plus *d* times *x* plus *e* times *y* plus *f* equals 0
$ax^2 + bxy + cy^2 + dx + ey + f = 0$.

---

25. lambda squared plus mi squared plus ni squared equals 1
$\lambda^2 + \mu^2 + \nu^2 = 1$.

---

26. *a* times *x* squared plus *b* times *y* squared plus *c* times *z* squared plus *d* times *x* times *y* plus *e* times *x* times *z* plus *f* times *y* times *z* plus *p* times *x* plus *q* times *y* plus *r* times *z* plus *s* equals 0
$ax^2 + by^2 + cz^2 + dxy + exz + fyz + px + qy + rz + s = 0$.

---

27. The scalar product of vector **i** times itself equals the scalar product of vector **j** times itself equals … equals 1
$\mathbf{i} \circ \mathbf{i} = \mathbf{j} \circ \mathbf{j} = \mathbf{k} \circ \mathbf{k} = 1$.

---

28. Vector **a** equals scalar *a x* times **i** plus scalar *a y* times **j** plus scalar *a z* times **k**
$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$.

---

29. Vector **b** equals scalar *b x* times **i** plus scalar *b y* times **j** plus scalar *b z* times **k**
$\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$.

---

30. The cosine of the angle between vectors **a** and **b** is equal to their scalar product divided by a product of their lengths

$$\cos\angle(\mathbf{a},\mathbf{b}) = \frac{\mathbf{a}\circ\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{\left(a_x^2 + a_y^2 + a_z^2\right)\left(b_x^2 + b_y^2 + b_z^2\right)}}\,.$$

---

31. The scalar product of vectors **a** and **b**, where **a** and **b** have coordinates *a* 1 to *a n* and *b* 1 to *b n* respectively is the sum of products *a* 1 times *b* 1 to *a n* times *b n* and is equal to the scalar product of vectors **b** and **a**
$\mathbf{a}\circ\mathbf{b} = \left(a_1, a_2, \ldots, a_n\right)\cdot\left(b_1, b_2, \ldots, b_n\right) = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \mathbf{b}\circ\mathbf{a}$.

---

32. The scalar product of vectors **a** and the sum of **b** and **c** is the scalar product of **a** and **b** plus the scalar product of **a** and **c**
$\mathbf{a}\circ\left(\mathbf{b}+\mathbf{c}\right) = \mathbf{a}\circ\mathbf{b} + \mathbf{a}\circ\mathbf{c}$.

---

33. The scalar product of vectors **a** and **b** multiplied by scalar *d* is equal to the scalar product of **a** times *d* and **b** and is also equal to the scalar product of **a** and **b** times *d*
$d(\mathbf{a}\circ\mathbf{b}) = \left(d\mathbf{a}\right)\circ\mathbf{b} = \mathbf{a}\circ\left(d\mathbf{b}\right)$.

---

34. The length of **a** is the sum of its squared coordinates
$|\mathbf{a}| = a_1^2 + a_2^2 + \cdots + a_n^2$.

---

35. The length of the sum of vectors **x** 1 to **x** *n* is less or equal than the sum of the lengths of these vectors
$\left|\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_n\right| \le \left|\mathbf{x}_1\right| + \left|\mathbf{x}_2\right| + \cdots + \left|\mathbf{x}_n\right|$.

---

36. The absolute value of the scalar product of *a* and *b* is less or equal than the product of the lengths of these vectors
$$\left|\mathbf{a}\circ\mathbf{b}\right| \le |\mathbf{a}||\mathbf{b}| = \sum_{i=1}^{n} a_i b_i \le \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2\ .$$

---

37. The cosine of the angle between vectors **a** and **b** is equal to their scalar product divided by the product of their lengths and the angle is between 0 and pi
$$\cos\angle(\mathbf{a},\mathbf{b}) = \frac{\mathbf{a}\circ\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\,,\quad 0 \le \angle(\mathbf{a},\mathbf{b}) \le \pi\,.$$

---

38. 2 *x* minus *y* divided by 4 plus 8 equals 0
$2x - y/4 + 8 = 0$.

---

39. 2 minus 3 divided by 4 plus 8 is not equal to 0
$2 - 3/4 + 8 \ne 0$.

---

40. Point *P* 1 has coordinates 57 and 88
$P_1(57, 88)$.

---

41. The line given by the equation *y* minus 8 equals 2 times, open brackets, *x* minus 17, close brackets
$y - 8 = 2\,(x - 17)$.

---

42. *d* equals the length of segment *AR*, that is two point four seven three
$d = |AR| = 2.473$.

---

43. Angle tau, that is the angle at vertex *A* of triangle *RAS* is 42 degrees 26 minutes and 10 seconds
$\tau = \sphericalangle RAS = 42°26'10''$.

---

44. *L* cubed (where *L* is a set) is the Cartesian product of *L* times *L* times *L*
$L^3 = L \times L \times L$.

---

45. *V* equals 4 divided by 3 times pi times *r* cubed (*V* is the volume of a sphere with radius *r*)
$V = 4/3\pi r^3$.

---

46. *S* equals 4 times pi times *r* squared (*S* is the surface of the sphere with radius *r*)
$S = 4\pi r^2$.

---

47. The probability of event *A* equals 365 times 364 times … times 365 minus *N* plus 1 all divided by 365 to *N*-th power
$P(A) = 365\cdot 364\cdot 363\cdot \ldots \cdot(365 - N + 1)/365^N$.

---

48. The binomial coefficient 90 over 5 equals 43 million 949 thousand 268

$$\binom{90}{5} = 43\ 949\ 268.$$

---

49. 5 over 4 times 90 –5 over 1 equals 5 over 4 times 85 over 1 equals 5 times 85 equals 425

$$\binom{5}{4}\binom{90-5}{1} = \binom{5}{4}\binom{85}{1} = 5 \cdot 85 = 425\,.$$

---

50. *a n* of *x* times the *n*-th order derivative of function *y* of *x* plus *a n*–1 of *x* times the *n*–1 order derivative of *y* plus … plus *a* 1 of *x* times the derivative of *y* plus a 0 of *x* times *y* equals function *f* of argument *x*

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_1(x)\frac{dy}{dx}+a_0(x)y=f(x)\,.$$

---

51. *h* is a function of argument *y* which equals the inverse of *y*
    $h(y) = 1/y$.

---

52. The derivative of function *x* with respect to *y* equals *f*
    $\frac{\mathrm{d}x}{\mathrm{d}y}=f$.

---

53. *y* equals the square root of eta squared plus 2 times open brackets *x* minus xi close brackets – for *x* greater than xi minus eta squared divided by 2
    $y=\sqrt{\{\eta^2+2(x-\xi)\}}$ for $x>\xi-\eta^2/2$.

---

54. *s* is an element of the union of set *S* and the set consisting of element *e*
    $s\in S\cup\{e\}$.

---

55. Permutation *p* 1 maps 1 to 2, 2 to 3, 3 to 1 and 4 to 4; permutation *p* 2 maps 1 to 4, 2 to 1, 3 to 3 and 4 to 2
    $p_1=\begin{pmatrix}1&2&3&4\\2&3&1&4\end{pmatrix}$ and $p_2=\begin{pmatrix}1&2&3&4\\4&1&3&2\end{pmatrix}$.

---

56. The product of permutations *p* 1 and *p* 2 is a permutation that maps 1 to 1, 2 to 3, 3 to 4 and 4 to 2
    $p_1\cdot p_2=\begin{pmatrix}1&2&3&4\\1&3&4&2\end{pmatrix}$.

---

57. The partial derivative of function *u* with respect to *x* equals zero
    $\frac{\partial u(x,y)}{\partial x}=0$.

---

58. Function *u* of arguments *x* and *y* equals function *f* of argument *y*
    $u(x,y)=f(y)$.

---

59. The derivative of function *u* with respect to *x* equals zero
    $\frac{du}{dx}=0$.

---

60. Function *u* of argument *x* equals *c*
$u(x) = c$.

---

61. The sum of partial second-order derivatives of function *u* with respect to *x* and *y*
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

---

62. Function *u* of arguments *x* and *y* equals zero whenever *y* equals zero
$u(x,0) = 0$.

---

63. The partial derivative of function *u* with respect to *y* at (*x*,0) equals the sine of *n* times *x* divided by *n*
$\frac{\partial u}{\partial y}(x,0) = \frac{\sin nx}{n}$.

---

64. Function *u* of arguments *x* and *y* equals the hyperbolic sine of *n y* times the sine of *n x* all divided by *n* squared
$u(x,y) = \frac{(\sinh ny)(\sin nx)}{n^2}$.

---

65. Both sides of this equation represent different ways of denoting the partial derivative of function *u* with respect to *x*
$u_x = \frac{\partial u}{\partial x}$.

---

66. Each of these expressions represents a different way of denoting the second order partial derivative of function *u* with respect to *x* and *y*
$u_{xy} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$.

---

67. This equation shows the definition of the nabla symbol as a gradient of a function – that is a vector consisting of a function's partial derivatives with respect to all its arguments
$\nabla = (\partial_x, \partial_y, \partial_z) f$.

---

68. The partial derivative of function *u* with respect to *t* equals alpha times the second-order derivative of function *u* with respect to *x*
$u_t = \alpha u_{xx}$.

In the following equations let *c* denote the inverse of square root of two times pi

69. Function *u* of arguments *t* and *x* equals ***c*** times the integral of function *F* of argument ksi times the exponential function of the argument minus alpha times ksi squared times *t* and times another exponential function of argument ksi times *x* multiplied by an imaginary unit – ksi is the so called “dummy variable” of integration in this case and the domain of integration is the interval from minus to plus infinity

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\xi) e^{-\alpha\xi^2 t} e^{-ix\xi} d\xi.$$

---

70. Function *f* of argument ksi equals ***c*** times the integral of function f of argument *x* multiplied by the exponential function of minus ksi times *x* multiplied by an imaginary unit; the integration variable is *x* and the domain of integration is an interval from minus to plus infinity

$$F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ix\xi} dx \, .$$

---

71. *F* is a constant function equal to ***c***

$$F(\xi) = \frac{1}{\sqrt{2\pi}} \, .$$

---

72. Function u of arguments *t* and *x* equals c squared times the integral of the exponential function of minus alpha times ksi squared times *t* multiplied by the exponential function of argument ksi times *x* multiplied by an imaginary unit; the integration variable is ksi and the domain of integration is the interval from minus to plus infinity

$$u(t,x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-\alpha\xi^2 t} e^{-ix\xi} d\xi \, .$$

---

73. *y* equals *x* cubed plus one; *x* is strictly between minus infinity and plus infinity; *y* is strictly between minus infinity and plus infinity
y equals square root of *x* minus five; *x* is greater than or equal to five; *y* is greater than or equal to zero
y equals sine squared of *x* divided by absolute value of *x* minus 4; *x* different than four; *y* is greater than or equal to zero
R is a real number.

$y = x^3 + 1$ the function domain: $x \in \mathbf{R}$
the function codomain: $y \in \mathbf{R}$

$y = \sqrt{x-5}$ the function domain: $x \ge 5$
the function codomain: $y \ge 0$

$y = \frac{\sin^2 x}{|x-4|}$ the function domain: $x \ne 4$

the function codomain: $y \geq 0$

---

74. *f* at *x*-two is greater than *f* at *x*-one
$f(x_2) > f(x_1)$.

---

75. Absolute value of *f* of *x* is less than or equal to *M*
$|f(x)| \leq M$.

---

76. *f* at a is equal to limit of *f* of *x* as *x* tends to *a*
$f(a) = \lim_{x \to a} f(x)$.

---

77. *f* at minus *x* is equal to *f* at
x $f(-x) = f(x)$.

---

78. Sine of pi over two plus *P* is equal to sine of pi over two is equal to one
$\sin(\pi/2 + P) = \sin \pi/2 = 1$.

---

79. Sine of two times *x* is equal to sine of two times *x* plus 2 times pi times *n* is equal to sine of two times the sum of *x* and pi times *n*
$\sin 2x = \sin(2x + 2\pi n) = \sin(2[x + \pi n])$.

---

80. Ordered pair of numbers minus *b* over two times *a* and *c* minus *b* squared over four times *a*
$\left(-\frac{b}{2a}, \quad c - \frac{b^2}{4a}\right)$.

---

81. *y* is equal to *x* to the power one over two is equal to square root of *x*
y is equal to *x* to the power one over four is equal to fourth root of *x*
$y = x^{1/2} = \sqrt{x}$, $y = x^{1/4} = \sqrt[4]{x}$.

---

82. *y* is equal to plus-minus square root of *x*
$y = \pm\sqrt{x}$.

---

83. *y* is equal to logarithm in the base *a* of *x*
$y = \log_a x$.

---

84. For *y* equal tangent of *x* such that *x* is different than pi over two plus pi times *k*, where *k* is equal to zero, plus-minus one, plus minus two and so on, and *y* is strictly between minus and plus infinity

for *y* equal cotangent of *x* such that *x* is different than pi times *k*, where *k* is equal to zero, plus-minus one, plus minus two and so on, and *y* is strictly between minus and plus infinity.

for $y = \tan y$: $x \neq \pi/2 + \pi k$, $k = 0$, ± 1, $\pm 2, \ldots$; $-\infty < y < +\infty$

for $y = \cot y$: $x \neq \pi k$, $k = 0$, ± 1, $\pm 2, \ldots$; $-\infty < y < +\infty$

---

85. *x* squared equals *d* squared plus *e* squared minus two times *d* times *e* times cosine of tau equals six point four nine seven three one three; so *x* equals two point five four nine miles
$x^2 = d^2 + e^2 - 2de \cos \tau = 6.497313$, so $x = 2.549$ miles.

---

86. Sine of epsilon equals *e* over *x* times sine of tau; epsilon-one is equal to eighty three degrees twenty minutes and zero seconds; epsilon-two is equal to ninety six degrees forty minutes and zero secondssin and
$\varepsilon = (e/x) \sin \tau$. So $\varepsilon_1 = 83°20'00''$ or $\varepsilon_2 = 96°40'00''$.

---

87. Delta is equal to one hundred and eighty degrees minus epsilon plus tau; delta-one is equal to fifty four degrees thirteen minutes and fifty seconds; delta-two is equal to forty degrees fifty three minutes and fifty seconds
$\delta = 180° - (\varepsilon + \tau)$. So $\delta_1 = 54°13'50''$ or $\delta_2 = 40°53'50''$.

---

88. Tangent of epsilon minus delta divided by two equals *e* minus *d* divided by *e* plus *d* times the tangent of *e* plus delta divided by two
$$\tan\frac{\varepsilon-\delta}{2} = \frac{e-d}{e+d}\tan\frac{\varepsilon+\delta}{2}.$$

---

89. Epsilon plus delta equals one hundred and eighty degrees minus tau equals one hundred and thirty seven degrees thirty three minutes fifty seconds
$\varepsilon + \delta = 180° - \tau = 137°33'50''$.

---

90. Epsilon plus delta divided by two equals sixty eight degrees forty six minutes fifty five seconds
$(\varepsilon + \delta)/2 = 68°46'55''$.

---

91. Epsilon minus delta divided by two equals twenty seven degrees fifty three minutes eighteen seconds
$(\varepsilon - \delta)/2 = 27°53'18''$.

---

92. Epsilon is equal to ninety six degrees forty minutes and thirteen seconds delta is equal to forty degrees fifty three minutes thirty seven seconds
$\varepsilon = 96°40'13''$ and $\delta = 40°53'37''$.

---

93. *x* equals *e* times sine of tau over sine of epsilon equals two point five four nine miles
$$x = e\frac{\sin\tau}{\sin\varepsilon} = 2.549 \text{ miles}.$$

---

94. *d x* by *d y* equals *y*

$$\frac{dx}{dy} = y \, .$$

---

95. *x* equals xi plus integral from eta to *y* of *y d y* equals xi plus 1 over two times *y* squared minus eta squared

$$x = \xi + \int_{\eta}^{y} y dy = \xi + \frac{1}{2}(y^2 - \eta^2) \text{ for y > 0.}$$

---

96. *y* equals square root of eta squared plus two times *x* minus xi; for *x* greater than xi minus eta squared over two

$$y = \sqrt{\{\eta^2 + 2(x - \xi)\}} \text{ for } x > \xi - \eta^2/2.$$

---

97. *e* to the power of *i* times theta equals cosine of theta plus *i* times sine of theta

$$e^{i\theta} = \cos\theta + i\sin\theta \, .$$

---

98. *e* o the power of *i* times pi plus one equals zero

$$e^{i\pi} + 1 = 0 \, .$$

---

99. Function zeta of two equals the sum from *n* equal one to infinity of one divided by *n* squared

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6} \, .$$

---

100. Gamma is a limit of a sum 1 plus 1 over 2 plus 1 over 3 plus one over 4 plus and so on plus 1 over *n* minus natural logarithm of *n* as *n* goes to plus infinity

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln(n)\right).$$

Additional Material

I. As an integral part of the course the authors suggest that students should prepare presentations in small groups. The library of the Gdańsk University of Technology could be a good source of up-to-date materials. You can find on-line texts at www.bg.edu.pl

II. When you are left with some free time at the end of a lesson you can do some of the following exercises with stronger groups: (this can be done as a competition against time when students are divided into smaller groups. The group that is the first one to collect all the answers wins).

SCAN the texts and find a mathematician/mathematicians who:

1) was a child prodigy,
2) made his/her first ground-breaking discoveries while still a teenager,
3) supplied significant portions of the mathematics infrastructure required for quantum mechanics and general relativity,
4) laid the foundations of modern chaos theory,
5) is known as the “father of Algebra”,
6) developed lung cancer due to poor living conditions,
7) earned some money from playing chess,
8) introduced the mathematical theory of probabilities,
9) played a mayor role in the European politics and diplomacy of his day,
10) wrote the book entitled “The Doctrine of Chances”,
11) was the first computer scientist and information theorist,
12) tried to cheat and signed a deal with another scientist,
13) argued that light is composed of particles,
14) was murdered by his own sister,
15) had to give up his plans for a military career due to poor eyesight and switched to mathematics instead.

III. Prepare a panel discussion:
Ask students to work in groups of four. Each of them reads one biography. They should then work for a few minutes preparing arguments for “their” mathematician being the greatest. Then in their small groups they have to try to convince their opponents. This can also be done as a whole class discussion

IV. In groups of three go through the biographies and prepare 10 questions. Quiz another group.

Hypatia of Alexandria

An imagined portrait of Hypatia of Alexandria

Hypatia of Alexandria (in Greek: Υπατία) (c. 370–415) was a popular Hellenized Egyptian female philosopher, mathematician, astronomer/astrologer, and teacher who lived in Alexandria, in Hellenistic Egypt, and who contributed greatly to that city's intellectual community. Several works are attributed to her by later sources, including commentaries on Diophantus's Arithmetica, on Apollonius's Conics and on Ptolemy's works, but none have survived. Letters written to her by her pupil Synesius give an idea of her intellectual milieu. She was of the Platonic school, although her adherence to the writings of Plotinus, the 3rd century follower of Plato and principal of the neo-Platonic school, is merely assumed. Hypatia's contributions to science are reputed (on scant evidence) to include the invention of the astrolabe and the hydrometer.

Gerolamo Cardano

Gerolamo Cardano or Jerome Cardan or Girolamo Cardan (September 24, 1501 – September 21, 1576) was a celebrated Italian Renaissance mathematician, physician, astrologer, and gambler.

Today, he is best known for his achievements in algebra. He published the solutions to the cubic and quartic equations in his 1545 book "Ars magna". The solution to one particular case of the cubic, $x^3 + ax = b$ (in modern notation), was communicated to him by Niccolo Fontana Tartaglia (who later claimed that Cardano had sworn not to reveal it, and engaged Cardano in a decade-long fight), and the quartic was solved by Cardano's student Lodovico Ferrari. Both were acknowledged in the foreword of the book, as well as in several places within its body. In his exposition, he acknowledged the existence of what are now called imaginary numbers, although he did not understand their properties.

Cardano was notoriously short of money and kept himself afloat by being an accomplished gambler and chess player. His book about games of chance, "Liber de ludo aleae", written in the 1560s but published only in 1663 after his death, contains the first systematic treatment of probability, as well as a section on effective cheating methods.

Cardano invented several mechanical devices including the combination lock, the gimbal consisting of three concentric rings allowing a supported compass or gyroscope to rotate freely, and the Cardan shaft with universal joints, which allows the transmission of rotary motion at various angles and is used in vehicles to this day. He made several contributions to hydrodynamics and held that perpetual motion is impossible, except in celestial bodies. He published two encyclopedias of natural science which contain a wide variety of inventions, facts, and occult superstitions. He also introduced the Cardan grille, a cryptographic tool, in 1550.

Lodovico Ferrari

Lodovico Ferrari (February 2, 1522 – October 5, 1565) was an Italian mathematician.

He began his career as the servant of Gerolamo Cardano. He was extremely bright, so Cardano started teaching him mathematics. Ferrari aided Cardano on his solutions for quadratic equation and cubic equations, and was mainly responsible for the solution of quartic equations that Cardano published. While still in his teens, Ferrari was able to obtain a prestigious teaching post after Cardano resigned from it and recommended him. Ferrari eventually retired young (only 42) and quite rich. He then moved back to his home town to take up a professorship of mathematics in 1565. Shortly thereafter, he died of white arsenic poisoning, allegedly murdered by his sister.

René Descartes

René Descartes (March 31, 1596 – February 11, 1650), also known as Cartesius, was a noted French philosopher, mathematician, and scientist. Dubbed the "Founder of Modern Philosophy" and the "Father of Modern Mathematics," he ranks as one of the most important and influential thinkers of modern times. For good or bad, much of subsequent western philosophy is a reaction to his writings, which have been closely studied from his time down to the present day. Descartes was one of the key thinkers of the Scientific Revolution in the Western World. He is also honoured by having the Cartesian coordinate system used in plane geometry and algebra named after him.

Descartes frequently contrasted his views with those of his predecessors. In the opening section of the Passions of the Soul, he goes so far as to assert that he will write on his topic "as if no one had written on these matters before". Nevertheless many elements of his philosophy have precedents in late Aristotelianism, the revived Stoicism of the 16th century, or in earlier philosophers like Augustine. In his natural philosophy, he differs from the Schools on two major points: first, he rejects the analysis of corporeal substance into matter and form; second, he rejects any appeal to ends (divine or natural) in explaining natural phenomena. In his theology, he insists on the absolute freedom of God's act of creation.

Descartes founded 17th century continental rationalism, later advocated by Baruch Spinoza and Gottfried Leibniz, and opposed by the empiricist school of thought, consisting of Hobbes, Locke, Berkeley & Hume. Leibniz, Spinoza and Descartes were all versed in mathematics as well as philosophy, and Descartes and Leibniz contributed greatly to science as well. As the inventor of the Cartesian coordinate system, Descartes founded analytic geometry, that bridge between algebra and geometry crucial to the invention of the calculus and analysis. Descartes's reflections on mind and mechanism began the strain of western thought that much later, impelled by the invention of the electronic computer and by the possibility of machine intelligence, blossomed into, e.g., the Turing test. It is commonly believed that his most famous statement is Cogito ergo sum (French: *Je pense, donc je suis* or in English: *I think, therefore I am*), found in §7 of Principles of Philosophy (Latin) and part IV of Discourse on Method (French).

Mathematical legacy

Descartes's theory provided the basis for the calculus of Newton and Leibniz, by applying infinitesimal calculus to the tangent problem, thus permitting the evolution of that branch of modern mathematics. This appears even more astounding considering that the work was just intended as an example to his "Discours de la méthode pour bien conduire sa raison, et chercher la verité dans les sciences" (Discourse on the Method to Rightly Conduct the Reason and Search for the Truth in Sciences, known better under the shortened title "Discours de la méthode").

Descartes' rule of signs is also a commonly used method in modern mathematics to determine possible quantities of positive and negative zeros of a function. This method is taught in both Algebra II and Precalculus today.

Descartes also made contributions in the field of optics; for instance, he showed by geometrical construction using the Law of Refraction that the angular radius of a rainbow is 42° (i.e. the angle subtended at the eye by the edge of the rainbow and the ray passing from the sun through the rainbow's centre is 42°).

Pierre de Fermat

Pierre de Fermat (pronounced Fair-mah's) (August 17, 1601 – January 12, 1665) was a French lawyer at the Parlement of Toulouse, southwestern France, and a mathematician who is given credit for his contribution towards the development of modern calculus. With his insightful theorems Fermat created the modern theory of numbers. The depth of his work can be gauged by the fact that many of his results were not proved for over a century after his death, and one of them, the Last Theorem, took more than three centuries to prove.

By the time he was 30, Pierre was a civil servant whose job was to act as a link between petitioners from Toulouse to the King of France and an enforcer of royal decrees from the King to the local people. Evidence suggests he was considerate and merciful in his duties.

Since he was also required to act as an appeal judge in important local cases, he did everything he could to be impartial. To avoid socializing with those who might one day appear before him in court, he became involved in mathematics and spent as much free time as he could in its study. He was so skilled in the subject that he could be called a professional amateur.

He was mostly isolated from other mathematicians, though he wrote regularly to two English mathematicians, Digby and Wallis. He also corresponded with French mathematician, Father Mersenne who was trying to increase discussion and the exchange of ideas among French mathematicians. One was Blaise Pascal who, with Fermat, established a new branch of math – probability theory.

Besides probability theory, Fermat also helped lay the foundations for calculus, an area of math that calculates the rate of change of one quantity in relation to another, for example velocity and acceleration. In particular, he is the precursor of differential calculus with his method of finding the greatest and the smallest ordinates of curved lines, analogous to that of the then unknown differential calculus.

Fermat himself was secretive and, since he rarely wrote complete proofs or explanations of how he got his answers, was mischievously frustrating for others to understand. He loved to announce in letters that he had just solved a problem in math but then refused to disclose its solution, leaving it for others to figure out.

Fermat's passion in math was in yet another branch – number theory, the relationship among numbers. While he was studying an ancient number puzzle book, he came up with a puzzle of his own that has been called Fermat's Enigma. Mathematicians worked for over three centuries to find its answer, but no one succeeded until 1994. Andrew Wiles, an English mathematician, created a proof and published it 330 years after Fermat's death in 1665.

Although he carefully studied and drew inspiration from Diophantus, Fermat inaugurated a different tradition. Diophantus was content to find a single solution to his equations, even if it was a fraction. Fermat was only interested in integer solutions to his diophantine equations and he looked for all solutions of the equation. He also proved that certain equations had no solution, an activity which baffled his contemporaries.

He studied Pell's equation and Fermat, perfect, and amicable numbers. It was while researching perfect numbers that he created Fermat's theorem.

He created the principle of infinite descent and Fermat's factorization method.

He created the two-square theorem, and the polygonal number theorem, which states that each number is a sum of 3 triangular numbers, 4 square numbers, 5 pentagonal numbers, ...

He was the first to evaluate the integral of general power functions. Using an ingenious trick, he was able to reduce this evaluation to summing geometric series. The formula that resulted was a key hint to Newton and Leibniz when they independently developed the fundamental theorems of calculus.

Although Fermat claimed to be able to prove all his arithmetical results, few of his proofs (if he had them) have survived. And considering that some of the results are so difficult (especially considering the mathematical tools at his disposal) many, including Gauss, believe that Fermat was unable to do so.

Together with René Descartes, Fermat was one of the two leading mathematicians of the first half of the 17^{th} century. Independently of Descartes, he discovered the fundamental principle of analytic geometry. Through his correspondence with Blaise Pascal, he was a co-founder of the theory of probability.

Blaise Pascal

Blaise Pascal (June 19, 1623 – August 19, 1662) was a French mathematician, physicist, and religious philosopher. Pascal was a child prodigy, who was educated by his father. Pascal's earliest work was in the natural and applied sciences, where he made important contributions to the construction of mechanical calculators and the study of fluids, and clarified the concepts of pressure and vacuum by expanding the work of Evangelista Torricelli. Pascal also wrote powerfully in defense of the scientific method.

Pascal helped create two major new areas of research. He wrote a significant treatise on the subject of projective geometry at the age of sixteen and corresponded with Pierre de Fermat from 1654 on probability theory, strongly influencing the development of modern economics and social science.

Following a mystical experience in late 1654, he left mathematics and physics and devoted himself to reflection and writing about philosophy and theology. His two most famous works date from this period: the Lettres provinciales and the Pensées. However, he had suffered from ill-health throughout his life and his new interests were ended by his early death two months after his 39^{th} birthday.

Contributions to mathematics

In addition to the childhood marvels recorded above, Pascal continued to influence mathematics throughout his life. In 1653 Pascal wrote his Traité du triangle arithmétique in which he described a convenient tabular presentation for binomial coefficients, the "arithmetical triangle", now called Pascal's triangle. (Yang Hui, a Chinese mathematician of the Qin dynasty, had independently worked out a concept similar to Pascal's triangle four centuries earlier.)

In 1654, prompted by a friend interested in gambling problems, he corresponded with Fermat on the subject, and from that collaboration was born the mathematical theory of probabilities. The friend was the Chevalier de Méré, and the specific problem was that of two players who want to finish a game early and, given the current circumstances of the game, want to divide the stakes fairly, based on the chance each has of winning the game from that point. (This was the introduction of the notion of expected value.) Pascal later (in the Pensées) used a probabilistic argument, Pascal's Wager, to justify belief in God and a virtuous life. The work done by Fermat and Pascal into the calculus of probabilities laid important groundwork for Leibniz's formulation of the infinitesimal calculus.

After a religious experience in 1654, Pascal mostly gave up work in mathematics. However, after a sleepless night in 1658 he offered, anonymously, a prize for the quadrature of a cycloid. Solutions were offered by Wallis, Huygens, Wren, and others; then Pascal, under a pseudonym, published his own solution. A controversy followed in which the competitors, including Pascal, behaved less than philosophically.

Isaac Newton

Sir Isaac Newton, (January 4, 1643 – March 31, 1727) was an English mathematician, physicist, astronomer, alchemist, chemist, inventor, and natural philosopher who is generally regarded as one of the greatest scientists and mathematicians in history.

Newton wrote the "Philosophiae Naturalis Principia Mathematica" in which he described universal gravitation and the three laws of motion, laying the groundwork for classical mechanics. By deriving Kepler's laws of planetary motion from this system, he was the first to show that the motion of objects on Earth and of celestial bodies are governed by the same set of natural laws. The unifying and deterministic power of his laws was integral to the scientific revolution and the advancement of heliocentrism.

Among other scientific discoveries, Newton realized that the spectrum of colours observed when white light passes through a prism is inherent in the white light and not added by the prism (as Roger Bacon had claimed in the thirteenth century), and notably argued that light is composed of particles. He also developed a law of cooling, describing the rate of cooling of objects when exposed to air. He enunciated the principles of conservation of momentum and angular momentum. Finally, he studied the speed of sound in air, and voiced a theory of the origin of stars. Despite this renown in mainstream science, Newton actually spent more time working on alchemy than physics, writing considerably more papers on the former than the latter.

Newton played a major role in the history of calculus, sharing credit with Gottfried Leibniz. He also made contributions to other areas of mathematics, for example the generalized binomial theorem. The mathematician and mathematical physicist Joseph Louis Lagrange (1736–1813), said that "Newton was the greatest genius that ever existed and the most fortunate, for we cannot find more than once a system of the world to establish."

Mathematical research

Newton became a fellow of Trinity College in 1669. In the same year he circulated his findings in "De Analysi per Aequationes Numeri Terminorum Infinitas" (On Analysis by Infinite Series), and later in "De methodis serierum et fluxionum" (On the Methods of Series and Fluxions), whose title gave rise to the "method of fluxions".

Newton and Gottfried Leibniz developed the calculus independently, using different notations. Although Newton had worked out his method years before Leibniz, he published almost nothing about it until 1693, and did not give a full account until 1704. Meanwhile, Leibniz began publishing a full account of his methods in 1684. Moreover, Leibniz's notation and "differential method" were universally adopted on the Continent, and after 1820 or so, in the British Empire. Newton claimed that he had been reluctant to publish his calculus because he feared being mocked for it. Starting in 1699, other members of the Royal Society accused Leibniz of plagiarism, and the dispute broke out in full force in 1711. Thus began the bitter calculus priority dispute with Leibniz, which marred the lives of both Newton and Leibniz until the latter's death in 1716. This dispute created a divide between British and Continental mathematicians that may have retarded the progress of British mathematics by at least a century.

Newton is generally credited with the generalized binomial theorem, valid for any exponent. He discovered Newton's identities, Newton's method, classified cubic plane curves (polynomials of degree three in two variables), made substantial contributions to the theory of finite differences, and was the first to use fractional indices and to employ coordinate geometry to derive solutions to Diophantine equations. He approximated partial sums of the harmonic series by logarithms (a precursor to Euler's summation formula), and was the first to use power series with confidence and to revert power series. He discovered new formulae for pi.

He was elected Lucasian professor of mathematics in 1669. In that day, any fellow of Cambridge or Oxford had to be an ordained Anglican priest. However, the terms of the Lucasian professorship required that the holder not be active in the church (presumably so as to have more time for science). Newton argued that this should exempt him from the ordination requirement, and Charles II, whose permission was needed, accepted this argument. Thus a conflict between Newton's religious views and Anglican orthodoxy was averted.

Gottfried Leibniz

Gottfried Wilhelm Leibniz (also Leibnitz or von Leibniz) (July 1, 1646 – November 14, 1716) was a German polymath of Polish origin (his great grandfather was called Lubieniecki).

Educated in law and philosophy, and serving as factotum to two major German noble houses (one becoming the British royal family while he served it), Leibniz played a major role in the European politics and diplomacy of his day. He occupies an equally large place in both the history of philosophy and the history of mathematics. He invented calculus independently of Newton, and his notation is the one in general use since. In philosophy, he is most remembered for optimism, i.e., his conclusion that our universe is, in a restricted sense, the best possible one God could have made. He was, along with René Descartes and Baruch Spinoza, one of the three great 17th century rationalists, but his philosophy also both looks back to the Scholastic tradition and anticipates logic and analysis.

Leibniz also made major contributions to physics and technology, and anticipated notions that surfaced much later in biology, medicine, geology, psychology, knowledge engineering, and information science. He also wrote on politics, law, ethics, theology, history, and philology, even occasional verse. His contributions to this vast array of subjects are scattered in journals and in tens of thousands of letters and unpublished manuscripts. To date, there is no complete edition of Leibniz's writings, and a complete account of his accomplishments is not yet possible.

Mathematician

Although the mathematical notion of function was implicit in trigonometric and logarithmic tables, which existed in his day, Leibniz was the first, in 1692 and 1694, to employ it explicitly, to denote any of several geometric concepts derived from a curve, such as abscissa, ordinate, tangent, chord, and the perpendicular (Struik 1969: 367). In the 18th century, "function" lost these geometrical associations.

Leibniz was the first to see that the coefficients of a system of linear equations could be arranged into arrays, now called matrices, which can be manipulated to find the solution of the system, if any. This method was later called Cramer's Rule. Leibniz's discovery of Boolean algebra and of symbolic logic was discussed in the preceding section.

A comprehensive scholarly treatment of Leibniz's mathematical writings has yet to be written, perhaps because Series 7 of the Academy edition is very far from complete.

Calculus

Leibniz is credited, along with Isaac Newton, with the discovery of infinitesimal calculus. According to Leibniz's notebooks, a critical breakthrough occurred on November 11, 1675, when he employed integral calculus for the first time to find the area under the function $y = x$. He introduced several notations used to this day, for instance the integral sign $\int$ representing an elongated *S*, from the Latin word summa and the d used for differentials, from the Latin word differentia. Leibniz did not publish any of his results until 1684. For an English translation of this paper, see Struik (1969: 271–84), who also translates parts of two other key papers by Leibniz on the calculus. The product rule of differential calculus is still called "Leibniz's rule."

Leibniz's approach to the calculus fell well short of later standards of rigor (the same can be said of Newton's). We now see a Leibniz "proof" as being in truth mostly a heuristic hodgepodge, mainly grounded in geometric intuition and an intuitive understanding of differentials. Leibniz also freely invoked mathematical entities he called infinitesimals, manipulating them freely in ways suggesting that they had paradoxical algebraic properties. George Berkeley, in a tract called "The Analyst" and elsewhere, ridiculed this and other aspects of the early calculus, pointing out that natural science grounded in the calculus required just as big of a leap of faith as theology grounded in Christian revelation.

The calculus as we now know it emerged in the 19^{th} century, thanks to the efforts of Cauchy, Riemann, Weierstrass, and others, who based their work on a rigorous notion of limit and on a precise understanding of the real numbers. Their work banished infinitesimals into the wilderness of obsolete mathematics (although engineers, physicists, and economists continued to use them). But beginning in 1960, Abraham Robinson showed how to make sense of Leibniz's infinitesimals, and how to give them algebraic properties free of paradox. The resulting nonstandard analysis can be seen as a great belated triumph of Leibniz's mathematical and ontological intuition.

From 1711 until his death, Leibniz's life was envenomed by a long dispute with John Keill, Newton, and others, over whether Leibniz had invented the calculus independently of Newton, or whether he had merely invented another notation for ideas that were fundamentally Newton's. Hall (1980) gives a thorough scholarly discussion of the calculus priority dispute.

Topology

Leibniz was the first to employ the term *analysis situs* (LL §27), later employed in the 19th century to refer to what is now known as topology. There are two takes on this situation. On the one hand, Mates (1986: 240), citing a 1954 paper in German by Freudenthal, argues as follows:

"Although for [Leibniz] the situs of a sequence of points is completely determined by the distance between them and is altered if those distances are altered, his admirer Euler, in the famous 1736 paper solving the Konigsberg Bridge Problem and its generalizations, used the term geometria situs in such a sense that the situs remains unchanged under topological deformations. He mistakenly credits Leibniz with originating this concept. ...it is sometimes not realized that Leibniz used the term in an entirely different sense and hence can hardly be considered the founder of that part of mathematics."

Hirano (1997) argues differently, quoting Mandelbrot (1977 : 419) as follows:
"...To sample Leibniz' scientific works is a sobering experience. Next to calculus, and to other thoughts that have been carried out to completion, the number and variety of premonitory thrusts is overwhelming. We saw examples in 'packing,'... My Leibniz mania is further reinforced by finding that for one moment its hero attached importance to geometric scaling. In "Euclidis Prota"..., which is an attempt to tighten Euclid's axioms, he states...: 'I have diverse definitions for the straight line. The straight line is a curve, any part of which is similar to the whole, and it alone has this property, not only among curves but among sets.' This claim can be proved today."

Thus Mandelbrot's well-known fractal geometry drew on Leibniz's notions of self-similarity and the principle of continuity: natura non facit saltus. We also see that when Leibniz wrote, in a metaphysical vein, that "the straight line is a curve, any part of which is similar to the whole..." he was anticipating topology by more than two centuries. As for "packing," Leibniz told to his friend and correspondent Des Bosses to imagine a circle, then to inscribe within it three congruent circles with maximum radius; the latter smaller circles could be filled with three even smaller circles by the same procedure. This process can be continued infinitely, from which arises a good idea of self-similarity. Leibniz's improvement of Euclid's axiom contains the same concept.

Information technology

Leibniz may have been the first computer scientist and information theorist. Early in life, he discovered the binary number system (base 2), the one subsequently employed on all computers, then revisited that system throughout his career. On Leibniz and binary numbers, see Couturat (1901: 473–78). Leibniz anticipated Lagrangian interpolation and algorithmic information theory. His calculus ratiocinator anticipated aspects of the universal Turing machine. In 1934, Norbert Wiener claimed to have found in Leibniz's writings a mention of the concept of feedback, central to Wiener's later cybernetic theory.

In 1671, Leibniz began to invent a machine that could execute all four arithmetical operations, gradually improving it over a number of years. This machine attracted fair attention and was the basis of his election to the Royal Society in 1673. A number of such machines were made during his years in Hanover, by a craftsman working under Leibniz's supervision. It was not an unambiguous success because it did not fully mechanize the operation of carrying. Couturat (1901: 115) reported finding an unpublished note by Leibniz, dated 1674, describing a machine capable of performing some algebraic operations.

Leibniz was groping towards hardware and software concepts worked out much later by Charles Babbage and Ada Lovelace, 1830–45. In 1679, while mulling over his binary arithmetic, Leibniz imagined a machine in which binary numbers were represented by marbles, governed by a rudimentary sort of punched cards. Modern electronic digital computers replace Leibniz's marbles moving by gravity with shift registers, voltage gradients, and pulses of electrons, but otherwise they run roughly as Leibniz envisioned in 1679. Davis (2000) discusses Leibniz's prophetic role in the emergence of calculating machines and of formal languages.

Guillaume de l'Hôpital

Guillaume François Antoine, Marquis de l'Hôpital (1661 – February 2, 1704) was a French mathematician. He is perhaps best known for the rule which bears his name for calculating the limiting value of a fraction whose numerator and denominator either both approach zero or both approach infinity.

L'Hôpital is commonly spelled as both "L'Hospital" and "L'Hôpital". The Marquis spelled his name with an 's'; however, the French language has since dropped the 's' (it was silent anyway) and added a circumflex to the preceding vowel.

L'Hôpital was born in Paris, France. He initially had planned a military career, but poor eyesight caused him to switch to mathematics. He solved the brachistochrone problem, independently of other contemporary mathematicians, such as Isaac Newton. He died in Paris.

He is also the author of the first known textbook on differential calculus, L'Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes. Published in 1696, the text includes the lectures of his teacher, Johann Bernoulli, in which Bernoulli discusses the indeterminate form 0/0. It is the method for solving such indeterminate forms through repeated differentiation that bears his name.

In 1694 he forged a deal with Johann Bernoulli. The deal was that L'Hopital paid Bernoulli 300 Francs a year to solve all problems for him and in return keep his mouth shut. In 1704, after L'Hopital's death, Bernoulli revealed the deal to the world. In 1922 texts were found that give support for Bernoulli.

Leonhard Euler

Leonhard Euler (pronounced ⟨oiler⟩) (April 15, 1707 Basel, Switzerland – September 18, 1783 St Petersburg, Russia) was a Swiss mathematician and physicist. He is considered to be the dominant mathematician of the 18^{th} century and one of the greatest mathematicians of all time; he is certainly among the most prolific, with collected works filling over 70 volumes.

Euler developed many important concepts and proved numerous lasting theorems in diverse areas of mathematics, from calculus to number theory to topology. In the course of this work, he introduced much of modern mathematical terminology, defining the concept of a function, and its notation, such as sin, cos, and tan for the trigonometric functions. It's also worth mentioning that he had numerous contacts with the scientists in Gdańsk.

Discoveries

Euler's work touched upon so many fields that he is often the earliest written reference on a given matter. Physicists and mathematicians often jest that often times a discovery or theorem is named after the “first person after Euler to discover it”. A list of his fundamental discoveries is bound to be incomplete – he can be said to have founded elementary analysis, graph theory, and many of the physical applications of mathematics now fundamental to civil, mechanical, electrical and aeronautical engineering. So the following examples are just an incomplete sampling.

Euler was the first to publish formulas with the constant e (also known as Euler’s constant), and showed the usefulness, consistency, and simplicity of defining the exponent of an imaginary number by means of the Euler’s formula

$$e^{i\theta} = \cos\theta + i\sin\theta .$$

which establishes the central role of the exponential function in elementary analysis, where virtually all functions are either variations of the exponential function or polynomials. This formula was called “the most remarkable formula in mathematics” by Richard Feynman (Lectures on Physics, p.I-22-10). Euler’s identity is a special case of this

$$e^{i\pi} + 1 = 0 .$$

Euler discovered quadratic reciprocity and proved that all even perfect numbers must be of Euclid’s form. He investigated primitive roots, found new large primes, and deduced the infinitude of the primes from the divergence of the harmonic series. This was the first breakthrough in this area in 2000 years, heralding the birth of the analytic number theory. His work on factoring whole numbers over the complexes marked the beginning of the algebraic number theory. Amicable numbers had been known for 2000 years before Euler, and in all that time only 3 pairs were discovered. Euler found 59 more.

With Daniel Bernoulli, Euler developed the Euler-Bernoulli beam equation that allows the calculation of stress in beams. Euler also deduced the Euler equations, a set of laws of motion in fluid dynamics, formally identical to the Navier-Stokes equations, explaining, among other phenomena, the propagation of shock waves.

Leonhard Euler:

- Elaborated the theory of higher transcendental functions by introducing the gamma function and the gamma density functions.
- Introduced a new method for solving 4^{th} degree polynomials.
- Proved Newton’s identities, Fermat’s little theorem, Fermat’s theorem on sums of two squares, and made distinct contributions to the Lagrange’s four-square theorem.
- Made contributions to combinatorics, the calculus of variations and difference equations.
- Created the theory of hypergeometric series, q-series and the analytic theory of continued fractions.
- Solved a multitude of diophantine equations. Introduced and studied the hyperbolic trigonometric functions.
- Calculated integrals with complex limits, which led (via Cauchy) to contour integration and complex analysis.
- Discovered the addition theorem for elliptic integrals.
- Invented the calculus of variations, including its most well-known result, the Euler-Lagrange equation.
- Proved the binomial theorem for binomials with real number exponents.

- Described numerous applications of Bernoulli's numbers, Fourier series, Venn diagrams, Euler numbers, e and pi constants, continued fractions and integrals.
- Discovered the infinite product and partial fraction representations of the trigonometric functions.
- Explicated logarithms of negative numbers.
- Integrated Leibniz's differential calculus with Newton's method of fluxions. Pioneered applications of calculus to physics.
- Co-discovered the Euler-Maclaurin formula which facilitates calculation of integrals, sums, and series.
- Published substantial contributions to the theory of differential equations.
- Defined a series of approximations which are used in computational mechanics. The most useful of these approximations is known as the Euler's method.
- Created the Latin square, which likely inspired Howard Garns' number puzzle SuDoKu.
- In number theory, Euler invented the totient function. The totient $\varphi(n)$ of a positive integer n is defined to be the number of positive integers less than or equal to n and coprime to n. For example, $\varphi(8) = 4$ since the four numbers 1, 3, 5 and 7 are coprime to 8. With this function Euler was able to generalize Fermat's little theorem.
- 1735: Euler reaffirmed his scientific reputation by solving the long-standing Basel problem:

  $$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

  where $\zeta(s)$ is the Riemann zeta function and also described how to evaluate the zeta function at any positive even number.
- 1735: Euler defined the Euler-Mascheroni constant useful for solution of differential equations:

  $$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln(n)\right)$$

  In geometry and algebraic topology, there is a relationship (also called the Euler's Formula) which relates the number of edges, vertices, and faces of a convex polyhedron. Given such a polyhedron, the sum of the vertices and the faces is always the number of edges plus two. i.e.: $F + V = E + 2$. The theorem also applies to any planar graph. For nonplanar graphs, there is a generalization: If the graph can be embedded in a manifold M, then $F - E + V = \chi(M)$, where χ is the Euler characteristic of the manifold, a constant which is invariant under continuous deformations. The Euler characteristic of a simply-connected manifold such as a sphere or a plane is 2. A generalization of Euler's formula for arbitrary planar graphs exists: $F - E + V - C = 1$, where C is the number of components in the graph.
- 1736: Euler solved, or rather proved insoluble, a problem known as the seven bridges of Königsberg, publishing a paper Solutio problematis ad geometriam situs pertinentis which was the earliest application of graph theory or topology.
- 1739: Euler wrote "Tentamen novae theoriae musicae", which was an attempt to combine mathematics and music; someone commented upon it that "for musicians it was too advanced in its mathematics and for mathematicians it was too musical."

Johann Carl Friedrich Gauss

Carl Friedrich Gauss (Gauß) (April 30, 1777 – February 23, 1855) was a German mathematician and scientist of profound genius who contributed significantly to many fields, including number theory, analysis, differential geometry, geodesy, magnetism, astronomy and optics. Sometimes known as "the prince of mathematicians", Gauss had a remarkable influence in many fields of mathematics and science and is ranked beside Euler, Newton and Archimedes as one of history's greatest mathematicians.

Gauss was a child prodigy, of whom there are many anecdotes pertaining to his astounding precocity while a mere toddler, and made his first ground-breaking mathematical discoveries while still a teenager. He completed Disquisitiones Arithmeticae, his magnum opus (great work), at the age of twenty-four. This work was fundamental in consolidating number theory as a discipline and has shaped the field to the present day.

Henri Poincaré

Henri Poincaré, photograph from the frontispiece of the 1913 edition of "Last Thoughts"

Jules Henri Poincaré (April 29, 1854 – July 17, 1912), generally known as Henri Poincaré, was one of France's greatest mathematicians and theoretical physicists, and a philosopher of science. Poincaré is often described as the last "universalist" (after Gauss) capable of understanding and contributing in virtually all parts of mathematics.

As a mathematician and physicist, he made many original fundamental contributions to pure and applied mathematics, mathematical physics, and celestial mechanics. He was responsible for formulating the Poincaré conjecture, one of the most famous problems in mathematics. In his research on the three-body problem, Poincaré became the first person to discover a chaotic deterministic system which laid the foundations of modern chaos theory.

Poincaré introduced the modern principle of relativity and was the first to present the Lorentz transformations in their modern symmetrical form. The Poincaré group was named after him. Poincaré discovered the remaining relativistic velocity transformations and recorded them in a letter to Lorentz in 1905. Thus he obtained perfect invariance of all of Maxwell's equations, the final step in the discovery of the theory of special relativity.

David Hilbert

David Hilbert (January 23, 1862, Wehlau, East Prussia – February 14, 1943, Göttingen, Germany) was a German mathematician, recognized as one of the most influential mathematicians of the 19th and early 20th centuries. He established his reputation as a great mathematician and scientist by inventing or developing a broad range of ideas, such as invariant theory, the axiomization of geometry, and the notion of Hilbert space, one of the foundations of functional analysis. Hilbert and his students supplied significant portions of the mathematic infrastructure required for quantum mechanics and general relativity. He is one of the founders of proof theory, mathematical logic, and the distinction between mathematics and metamathematics, and warmly defended Cantor's set theory and transfinite numbers. A famous example of his world leadership in mathematics is his 1900 presentation of a set of problems that set the course for much of the mathematical research of the 20th century.

Stefan Banach

Stefan Banach (March 30, 1892 in Kraków, Austria-Hungary now Poland – August 31, 1945 in Lwów, Soviet Union – occupied Poland), was an eminent Polish mathematician, one of the moving spirits of the Lwów School of Mathematics in pre-war Poland. He was largely self-taught in mathematics; his genius was accidentally discovered by Juliusz Mien and later Hugo Steinhaus.

When World War II began, Banach was President of the Polish Mathematical Society and a full professor of University of Lwów. Being a corresponding member of Academy of Sciences of the Ukrainian SSR, and otherwise on good terms with Soviet mathematicians, he was allowed to keep his chair despite the Soviet occupation, from 1939, of the city. Banach survived the subsequent brutal German occupation from July 1941 up to February 1944, earning a living by feeding lice with his blood in the Typhus Research Institute of Prof. Rudolf Weigl. His health declined during the occupation, and he developed lung cancer. After the war Lwów was incorporated into the Soviet Union, and Banach died there before he could be repatriated to Kraków, Poland. He is buried at the Lyczakowski Cemetery.

Nicolas Bourbaki

Nicolas Bourbaki is the common name under which a group of mainly French 20th century mathematicians wrote a series of books presenting an exposition of modern advanced mathematics, beginning in 1935. With the goal of founding all of mathematics on set theory, the group strove for utmost rigour and generality, creating some new terminology and concepts along the way.

While Nicolas Bourbaki is an invented personage, the Bourbaki group is officially known as the "Association des collaborateurs de Nicolas Bourbaki" ("association of collaborators of Nicolas Bourbaki”), which has an office at the École Normale Supérieure in Paris.

The group

Accounts of the early days vary, but original documents have now come to light. The founding members were all connected to the Ecole Normale Supérieure in Paris and included Henri Cartan, Claude Chevalley, Jean Coulomb, Jean Delsarte, Jean Dieudonné, Charles Ehresmann, René de Possel, Szolem Mandelbrot and André Weil. There was a preliminary meeting, towards the end of 1934 (the minutes are in the Bourbaki archives – for a full description of the initial meeting consult Liliane Beaulieu in the Mathematical Intelligencer). Jean Leray and Paul Dubreil were present at the preliminary meeting but dropped out before the group actually formed. Other notable participants in later days were Laurent Schwartz, Jean-Pierre Serre, Alexander Grothendieck, Samuel Eilenberg, Serge Lang and Roger Godement.

The original goal of the group had been to compile an improved mathematical analysis text; it was soon decided that a more comprehensive treatment of all of mathematics was necessary. There was no official status of membership, and at the time the group was quite secretive and also fond of supplying disinformation. Regular meetings were scheduled, during which the whole group would discuss vigorously every proposed line of every book. Members had to resign by age 50.

The atmosphere in the group can be illustrated by an anecdote told by Laurent Schwartz. Dieudonné regularly and spectacularly threatened to resign unless topics were treated in their logical order, and after a while others played on this for a joke. Godement’s wife wanted to see Dieudonné announcing his resignation, and so on one occasion while she was there Schwartz deliberately brought up again the question of permuting the order in which measure theory and topological vector spaces were to be handled, to precipitate a guaranteed crisis.

The name “Bourbaki” refers to a French general Charles Denis Sauter Bourbaki who was defeated in the Franco-Prussian War; it was adopted by the group as a reference to a student anecdote about a hoax mathematical lecture, and also possibly to a statue. It was certainly a reference to Greek mathematics, Bourbaki being of Greek extraction. It is a valid reading to take the name as implying a transplantation of the tradition of Euclid to a France of the 1930s, with soured expectations. (It is said that Weil’s wife Eveline supplied Nicolas. They married in 1937, she having previously been with de Possel, who unsurprisingly left the group.)

Glossary

A

abelian – abelowy
abelian group – grupa abelowa, grupa przemienna
abstract – abstrakcyjny, abstrakt
abstract algebra – algebra abstrakcyjna
account – konto, rachunek
acute – ostry
acute angle – kąt ostry
add – dodać
added – dodany, dodatkowy
adding – dodawanie
addition – dodanie, dodatek, dodawanie
additional – dodatkowy
additive – dodatek, addytywny
adjoining – przyległy, sąsiedni
aggregate – suma, łączny wynik
algebra – algebra
algebraic – algebraiczny
algorithm – algorytm
alternating – występujący na przemian, naprzemienny
alternatively – ewentualnie, alternatywnie
amount – ilość, liczba
analogous – analogiczny
analogue – odpowiednik, analog
analysis – analiza, badanie
analytic – analityczny
angle – kąt
antiderivative of a function – funkcja pierwotna
applicable – obowiązujący, właściwy, odnoszący się do
application – zastosowanie
applied – stosowany
apply – zastosować, obowiązywać
approximate – przybliżony, zaokrąglony
approximation – przybliżenie, przybliżona liczba
arbitrary – przypadkowy, dowolny
arc tangent – arcus tangens
area – obszar, rejon
argument – argument, zmienna niezależna
arithmetic – arytmetyka, arytmetyczny
arrangement – ułożenie, rozmieszczenie
arrow – strzałka
arrowhead – grot
association – towarzystwo, związek, stosunek
associative – asocjacyjny, skojarzeniowy, łączny
assume – przypuszczać, założyć, przybrać
assumption – przypuszczenie, założenie
asymptotic – asymptotyczny
attribute – przypisać
ax(e) – cięcie, redukcja
axis – oś
axiom – aksjomat, pewnik
axiomatic – aksjomatyczny, oczywisty

B

bar – pręt, sztaba, poprzeczka, z wyjątkiem czegoś, także nawias wartości bezwzględnej
base – podstawa, baza, zasada
basic – podstawowy, zasadniczy, elementarny
basis – podstawa, zasada
bend – zakręt, ugięcie
binary – dwójkowy, binarny
bisector – dwusieczna, symetralna
bit – kawałek, fragment, końcówka
boundary – granica
branch – gałąź, ramię, rozgałęzienie

C

calculation – obliczenie, kalkulacja
calculus – rachunek, rachunek różniczkowy
cancel – odwołać, anulować, cofnąć
cancelation (of minus signs) – odwołanie, unieważnienie
cartesian – kartezjański
category – kategoria,
circle – koło, okrąg
circular – kolisty, okrągły
circumference – obwód
cm = centimetre – centymetr
coding – kodowanie
codomain (of a function) – przeciwdziedzina (funkcji)
coefficient – współczynnik

collection – kolekcja, zbiór
combination – połączenie, powiązanie, kombinacja
combinatorial – kombinatoryczny
combinatorics – kombinatoryka
combine – połączyć, powiązać
compile – sporządzić, opracować,
complex – zespół, złożony, skomplikowany, zespolony
component – składnik
comprehensive – obszerny, wyczerpujący
computation – obliczenie, wyliczenie
computing – obliczanie, wyliczanie
concept – pojęcie
condition – warunek, status
cone – stożek
congruence – przystawanie, kongruencja
congruent – stosowny, odpowiedni, przystający
conic – stożkowy
conjecture – przypuszczenie, domysł, hipoteza
conjugate – sprzężony
conjunction – połączenie, spójnik, koniunkcja
connection – związek,
consolidate – utrwalić, wzmocnić, połączyć
constant – element stały, stała
constraint – ograniczenie, przymus
contribute – wnieść, przyczyniać się do
converse – odwrotność
conversely – odwrotnie
coordinate – współrzędna
corner – kąt, róg, narożnik
correspondence – odpowiedniość
cosine – kosinus
count – liczenie, obliczanie
couple – para, połączyć, skojarzyć
criterion, *pl.* criteria – kryterium
cross-section – przekrój
cube – sześcian, kostka
cubic – sześcienny
curve – krzywa, łuk
cylinder – walec

D

de = differential equation – równanie różniczkowe
decagon – dziesięciobok, dziesięciokąt
decimal – dziesiętny
decreasing (function) – (funkcja) malejąca
deductive – dedukcyjny
define – definiować, określać
definite – określony
definition – definicja, określenie
degree – stopień
denotation – oznaczenie
depth – głębokość
derivative – pochodna
describe – opisywać
description – deskrypt, opis
descriptive geometry – geometria wykreślna
design – konstrukcja, projekt, wzór
detail – szczegół
determinant – wyznacznik, determinant
determine – wyznaczać
diagonal – przekątna, diagonalna
diagram – wykres, diagram, schemat
diameter – średnica
difference – różnica,
differential – różniczka,
dimension – wymiar
direction – kierunek
direct proportionality – wprost proporcjonalność
discontinuity – nieciągłość
discrete – dyskretny
discriminant – wyróżnik
disjoint – rozłączny
disjunction – alternatywa, suma logiczna, dysjunkcja
display – przedstawiać, obrazować
disquisition – rozprawa, wywód
distance – odległość
distribute – rozdzielać
diverse – wieloraki, różnorodny
divide – dzielić
division – dzielenie
domain (of a function) – dziedzina (funkcji)
dot – kropka, punkt
double – podwajać
drawing – rysunek, wykres
dynamical – dynamiczny
dynamics – dynamika

E

edge – krawędź, ostrze
eigenvalue – wartość własna
eigenvector – wektor własny
electrical – elektryczny
electromagnetism – elektromagnetyzm
element – element
elementary – elementarny, jednostkowy
elicit – uzyskać, wywołać
ellipsoid – elipsoida
elliptic – eliptyczny
eminent – wybitny, znakomity, wyjątkowy
endpoint – punkt końcowy
enumeration – numeracja, przeliczalność
equation – równanie
equiangular – równokątny
equilateral – równoboczny

equivalence – równoważność, równowartość
equivalent – równoważny, równorzędny
estimate – szacować
evaluation – wyznaczanie wartości, wyliczenie
even – parzysty, równy
exhibit – eksponat, wystawiać
exponent – wykładnik, eksponent
exponential – wykładniczy
exterior – strona zewnętrzna
extremal point – punkt ekstremalny

F

field – pole, obszar, ciało
figure – figura
finite – skończony
fixed – ustalony, stały, utrwalony
form – kształt, forma
formula – wzór
formulate – sformułować, opracować
fractal – fraktal
fraction – ułamek
framework – kratownica, szkielet konstrukcji
function – funkcja
fundamental – podstawowy

G

geometric/geometrical – geometryczny
geometry – geometria
graph – diagram, wykres, graf
graphical – wykreślny, graficzny

H

height – wysokość
heptagon – siedmiokąt
hexagon – sześciokąt
hexagonal – sześciokątny
homogeneous – jednorodny, homogeniczny
homomorphism – homomorfizm
horizontal – poziomy
hyperbola – hiperbola
hypotenuse – przeciwprostokątna

I

identify – identyfikować,
identity – tożsamość, identyczność, jedynka
incalculable – nieobliczalny, nieprzewidywalny
inclination – pochylenie, nachylenie
include – włączać, zawierać
inclusive – łączny, włączny
incorporate – włączyć, zawierać w sobie, obejmować
indefinite – nieoznaczony
indefinite integral – całka nieoznaczona
independent – niezależny
induce – indukować, wzbudzać, wywoływać
inequality – nierówność
inference – wniosek, konkluzja
infer – wnioskować
infinite – nieskończony
infinitesimal – nieskończenie mały
infinity – nieskończoność
infix – infiks, wrostek
ingredient – składnik
initial – początkowy
input – wejście, wkład
insert – wkładać, wstawiać
instance – przykład, konkret
integer – liczba całkowita
integral – integralny, całkowy, całkowity
interconnect – łączyć wzajemnie
intersect – przecinać się
intersection – przecięcie
intersection point – punkt przecięcia
invariance – niezmienniczość
invariant – niezmiennik
inverse – odwrotność
invertible – odwracalny
irregular – nieregularny, nieprawidłowy
isolate – wyodrębnić, oddzielić
isomorphic – izomorficzny
isomorphism – izomorfizm, równopostaciowość
isosceles – równoramienny

J

justified – uzasadniony
juxtaposition – zestawienie

L

lattice – krata, struktura
law – prawo, zasada
leap – skok, uskok
leap year – rok przestępny
limited – ograniczony
line – linia, prosta
linear – liniowy, linearny
linearity – liniowość
link – połączenie, linia
location – lokalizacja, usytuowanie
logic – logika, układ logiczny

M

magnetism – magnetyzm,
magnitude – wielkość, rozmiar, wartość bezwzględna
matric – macierzowy
martic equation – równanie macierzowe
matrix – macierz
mean – (wartość) średnia

measure – mierzyć, miara
measurement – pomiar
method – metoda
minus – minus
module – moduł
monoid – monoid, półgrupa z jedynką
monotone – monotoniczny
multiplication – mnożenie

N

n-dimensional space – przestrzeń n-wymiarowa
n-tuple – n-krotny
nabla – (operator) nabla
negative – ujemny
nine-sided – dziewięcioboczny
non-commutative – nieprzemienny
non-homogeneous – niejednorodny
non-linear – nieliniowy
non-singular – nieosobliwy
non-zero – niezerowy
nonagon – dziewięciokąt
nonempty – niepusty
notation – oznaczenie, notacja, zapis
null – zero, wartość zerowa
number – liczba, numer
numerical – liczbowy, numeryczny
numerous – liczny

O

object – przedmiot
obtuse – rozwarty
octagon – ośmiokąt
odd – nieparzysty
operation – działanie
operator – operator
opposite – przeciwległy, przeciwny
optimal value – wartość optymalna
optimization – optymalizacja
order – rząd, porządek
orientation – kierunek, orientacja
orthogonal – ortogonalny, prostopadły

P

panel – tablica, płyta
parabola – parabola
paraboloid – paraboloida
parallel – równoległy
parallelogram – równoległobok
parenthesis – nawias okrągły
pentagon – pięciokąt
period (of a function) – okres funkcji
permutation – permutacja
perpendicular – prostopadły
plane – płaszczyzna
point – punkt
polygon – wielokąt
polyhedron – wielościan
polynomial – wielomian
power – potęga, wykładnik potęgi
precedence – prawo pierwszeństwa
precipitate – strącać, wytrącać się
predicate – predykat
prefix – przedrostek, prefiks
prime – liczba pierwsza
principle – zasada, reguła
prism – graniastosłup
probability – prawdopodobieństwo
proof – dowód, sprawdzenie
proportional – proporcjonalny
prove – dowodzić
pyramid – ostrosłup

Q

quadratic – kwadratowy
quadric – forma kwadratowa
quadrilateral – czworobok
quantitative – ilościowy
quantity – ilość
quantum – kwant
quasi-linear – pseudo-liniowy, quasi-liniowy
quaternion – kwaternion
question – pytanie
quotient – współczynnik, iloraz

R

radical – radykalny, radykał (ideału)
radius – promień (plural: radii, radiuses)
random – przypadkowy, losowy
ratio – proporcja, in/by a ratio of 60:40
rational – racjonalny, rozsądny, wymierny
ray – promień, półprosta
reading – odczyt
real – rzeczywisty, realny
rectangle – prostokąt
rectangular – prostokątny
rejoined – ponownie połączyć
relationship – relacje
relativity – względność
relevant – istotny, mający znaczenie
represent – reprezentować
research – badania (pl)
respect – szacunek, szanować, ze względu
respective – odpowiedni
result – rezultat, wynik
rhombus – romb (*pl.* -uses, -i also rhomb)
ring – kółko, krąg
root – pierwiastek

S

scalar – skalar
scalene – ukośny, nierównoboczny
segment – odcinek
set – zbiór
seven-sided – siedmioboczny
shape – kształt
side – strona, bok
significance – znaczenie
single – pojedynczy
singular – w liczbie pojedynczej
six-sided – sześcioboczny
size – wymiary, wielkość
skew – przekrzywiony, ustawiać ukosem, ukośny
skier – narciarz
slice – część, kawałek
slope – nachylenie
solid – bryła
solution – rozwiązanie
space – przestrzeń
specify – określić, zdefiniować
spectral – widmowy
sphere – sfera
square – kwadrat
statement – twierdzenie
statistics – statystyka
stem – pień, trzon
string – łańcuch (numeryczny)
structure – struktura
subset – podzbiór
subtract – odjąć, odejmować
sum – rachunek, obliczenie, suma
summarize – streścić, podsumować
supplement – uzupełniać, uzupełnienie
supplementary – dopełniający (eg. angles)
symbol – symbol
symmetrical – symetryczny
symmetry – symetria
system – system

T

tangent – tangens, styczny
tensor – tensor
terminology – nazewnictwo, terminologia
tetrahedron – czworościan
theorem – twierdzenie
theory – teoria
thermodynamics – termodynamika
thirteen-dimensional – trzynastowymiarowy
three-dimensional – trójwymiarowy
three-sided – trójboczny, trójstronny
total – całkowity, totalny
transform – przekształcić, transformować
transformation – przekształcenie
trapezoid – trapez
triangle – trójkąt
triangle-shaped – w kształcie trójkąta
triangular – trójkątny
trigonometrical – trygonometryczny
trigonometry – trygonometria
two-dimensional – dwuwymiarowy

U

unit – jednostka
unknown – niewiadoma

V

value – wartość, wielkość
variable (dependent/free/random) – zmienna (zależna/niezależna/losowa)
vector – wektor
vector space – przestrzeń wektorowa
velocity – prędkość
vertex – wierzchołek (*pl.* vertices, vertexes)
vertical – pionowy, pion
vertices, *pl.* of vertex – wierzchołki
volume – objętość

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Teacher's notes

Chapter 1

1. Ask students to discuss the definition of mathematics in pairs. Check with the whole group.

2. Ask students to listen to the recording and repeat the following words:
quantity, science, branch, space, equation, calculus, integer, topic, to solve, pattern, application.

3. Ask students to work in pairs and to discuss and explain the meanings of these words.

4. Ask students to read the text and find new words.

5. Ask students to close the books and make a list of useful terms they remember from the text. Then ask them to compare their lists in pairs ad discuss them.

6. Give students a few minutes to find the words. Check with the whole group.

Key: a) quantity, b) structure, c) calculus, d) vector, e) equation, f) geometry, g) number, h) function, i) fraction.

7. Ask students to quiz each other in groups of four: one person says the definition, the others give the word. Swap roles.

8. Ask students to find the plural forms of the given words
Key: quantity – quantities, branch – branches, calculus – calculuses/calculi, area – areas, science – sciences, axiom – axioms, article – articles, computation – computations, perspective – perspectives, consideration – considerations.

Ask students if they know the rules for forming plurals (revise them). Ask them to divide the nouns into groups according to the rules.

9. Ask students to try reading the given math operations. You can check some of them in the recorded material.

10. If time allows – ask each student to choose a biography of a famous mathematician from the "Additional material". Ask them to read the texts and explain the unknown words. Having finished that, they should prepare a short summary with the most important facts. When they are ready ask them to work in groups of three (with 3 different biographies in each group). They should summarize their biography for their partners.

Chapter 2

1. Books closed. Ask the students to discuss in pairs what basic mathematical operations they know.
   Elicit: *addition*, *subtraction*, *multiplication*, *division*.
2. Books closed. Elicit or pre-teach the following terms (writing symbols on the board): *sign*, *fraction*, *exponent*, *square root*, *arrow*, *decimal*, *parentheses*, *even*/*odd number*, *power*, *greater than*, *less than*, 2^3 (*two to the third power*), 4^2 (*four square*), *to cancel*, *to simplify*, *to multiply*, *to divide*, *to count*.
3. Books open. Ask students to repeat the words.
4. Ask students to read the text and find more useful words. When they finish reading ask them to compare in pairs what words they found. Get feedback with the whole group.
5. Ask students to write 10 questions to the text individually. Tell them to make sure they know the answers.
6. Ask students to work in groups of 3 or 4 and ask the questions. Allow students to refer to the text so that they practice vocabulary.
7. Ask students to write 4 mathematical operations and dictate them in pairs. They should compare when they finish.
8. Ask students to work in pairs. One of them says a word from the text in English, the other one gives the Polish equivalent. Tell them to swap roles after some time.
9. Before doing this exercise revise the rules of question formation (especially "Wh…" questions) with your students. Ask students to work in pairs A and B. Assign them text A (Appendix 1A) or B (Appendix 1B). Tell them to read their texts carefully and explain the unknown words. When they finish give them a few minutes to prepare questions about the gaps in their texts individually. Monitor the correctness of questions. Then tell them to work in pairs and ask each other the questions to obtain the information they don't have. When they finish check the text with the whole group(for example you could ask one of the students to read it aloud).

   You are given the whole text as a key:

Calculus was created in large part by Newton and Leibniz, although some of the ideas were already used by Fermat and even Archimedes. Calculus is divided into two parts, which are closely related. One part is called "differential calculus" and the other part is called "integral calculus".

Integral calculus is concerned with area and volume. How do you determine the area of a circle or the volume of a sphere? Another way of putting it is: how much paint do you need to color in a circle? How much water do you need to fill up a ball? Integral calculus explains one way of computing such things.

The basic idea of integral calculus is this: the simplest shape whose area we can compute is the rectangle. The area is the length of the rectangle multiplied by its width. For instance, a "square mile" is a piece of land with as much area as a square plot of land with sides measuring one mile each. To compute the area of a more complicated region, we chop up the region into lots and lots of little rectangles. When we do this, we will not be able to succeed completely because there will always be pieces with curved sides, generally. But the key idea is that the sum of the areas of the rectangular pieces will be a very close approximation of the actual area, and the more pieces we cut, the closer our approximation will be.

Differential calculus answers the following question: imagine you go on a car ride. Suppose you know your position at all times. In other words, at 10 a.m. you're in the garage, at 10 a.m. and 5 seconds you're just outside the garage, at 10 a.m. and 10 seconds you're on the road just in front of your house...and so on... At the end of your trip, you realize that at every moment during your trip, your speedometer showed the speed of your car. Just from the knowledge of your position at all times, can you reconstruct what your speedometer showed at any time? The answer is, yes, you can, and differential calculus provides a method for doing this.

Key A:

1. figures, 2. plane, 3. acorn, 4. true statements, 5. dry, 6. enumerative, 7. geometry, 8. Bell's day

Key B:

1. solid, 2. (dealing with objects like the spherical triangle and spherical polygon), 3. medieval, 4. accepted truths, 5. proof. 6. projective, 7. extensive, 8. has grown

Try reading aloud – ask students to try and read the operations in pairs. Direct them to Appendix 9 for extra practice.

Chapter 3

1. Books closed. Ask the students to discuss the definition of linear algebra in pairs. Get feedback with the whole group.

2. Books closed. Ask students to listen to the recording and repeat the words.

3. Ask students to read the text and find the words in the text.

4. Ask students to work out the definitions of the terms from exercise 2. Insist on definitions in English and make sure students write them down.

5. Ask students to read their definitions to the whole group and everybody will guess the words.

6. Ask students to discuss the exact Polish equivalents of the terms. Get feedback with the whole group.

7. Ask students to write true/false sentences individually or in pairs. Make sure they refer to the text. If they prepare the sentences in pairs they should read and discuss them in groups of four.

8. Ask students to match the given words with their definitions.

Key: 1. D, 2. A, 3. D, 4. C.

9. Ask students to form the plural forms of the following words:

derivative – derivatives	theory – theories
matrix – matrices	length – lengths
determinant – determinants	entity – entities
study – studies	*n*-tuple – *n*-tuples
analysis – analyses	basis – bases

10. Ask students to read the whole text carefully, read the phrases and put them back in the text

Key: 1) from one or more variables and constants,
2) important in all levels and all areas of,
3) the basic methods of manipulation,
4) physical sciences,
5) (called the coefficient),
6) equal to the degree of that variable,
7) a constant monomial,
8) irrational numbers,
9) constructed from,
10) in several variables,
11) the simultaneous zero sets of,
12) encode information,
13) information about the operator's eigenvalues,
14) the simplest algebraic,
15) may have infinite degree.

Chapter 4

1. Books closed. Ask students what analytic geometry is. Accept all ideas.

2. Books closed. Pre-teach the following words:
graph, *angle*, *equation*, *tangent*, *straight line*, *cosines*, *arrow*, *axes*, *slope*, *coordinate*, *steep line*, *variable*, *parabola*, *hyperbola*, *intersection*, *curve*, *square root*, *quadratic*.

3. Play the recording, ask the students to repeat the words.

4. Ask students to read the text and note down unknown words. Get feedback when they finish. Which words were difficult?

5. Ask students to read the definitions and match the words. Give a few minutes to complete the exercise. Check answers.

Key: 1) graph, 2) equation, 3) slope, 4) curve, 5) square root, 6) straight line, 7) steep line, 8) hyperbola

For further practice, ask students to work in pairs and quiz each other: one person reads the definitions-the other (book closed) gives the word. Swap roles.

6. Ask students to read the statements and decide individually whether they are true or false. Let them consult their partner.

Key: false, true, false, true

7. Ask students to read the text and put the words in the right places.

Check with the whole group.

Key: 1. coordinate, 2. geometry, 3. equations, 4. planes, 5. circles, 6. dimensions, 7. shapes, 8. numerical, 9. vector, 10. analytic

8. Ask students to read the text and prepare a brief summary in pairs. They should put 10 factual mistakes in it. Make sure students understand what factual means.

9. Ask them to get in groups of four. One pair reads the summary; the other pair should listen and find the mistakes. Ask them to check with the text. Swap roles.
10. If there is any time left, ask students to prepare 5 questions to the second text. Ask them to make groups of three, ask their questions and test each other.

Chapter 5

1. Ask students what a function is.
2. Ask students to repeat the terms after the recording:

   domain, *codomain*, *argument*, *correspondence*, *monotone*, *increasing*, *bounded*, *discontinuous.*
3. Ask students to read the text and find out more about the basic notions of functions
4. Ask students to write 6 true/false sentences to the text (individually).
5. Ask students work in pairs. They should read their sentences to their partner and ask them to decide if they are true or false. Tell them to refer to the text if necessary. Swap roles.
6. Ask students to work in groups of four. They should decide who is A, B, C, or D. Then they should read their texts and prepare short summaries.
7. Students should still work in their groups of 4. Each of them should read their summary to the group, explaining the important terms together.
8. Ask students to make a list of 10 important terms from this chapter. They should quiz one another in groups of three.
9. Ask students to work in pairs. On if them draws a graph, and the other person should try and name it. Then they should swap roles.
10. Ask students to close the books and ask the definitions for the following notions from memory:
    — domain of function,
    — linear function,
    — hyperbola,
    — quadratic function,
    — power function.

    They should check their definitions in pairs, and refer to the text if necessary.

Chapter 6

1. Ask students what geometrical terms they know.
2. Ask students to repeat the following words after the recording:
   line, *geometry*, *intersection*, *ray*, *endpoint*, *parallel*, *perpendicular*, *angle*, *vertex*, *degree*, *acute angle*, *obtuse angle*, *angle bisector*.
3. Ask the students to read the text and write down any new terms.
4. Ask the students to do the true/false statements. Let them check their answers in pairs.
   Key: a) true, b) false, c) false, d) true, e) false

5. Ask students to prepare five definitions of geometrical terms from this chapter. They should then work in pairs reading their definitions to their partners and asking them what the terms are. When they finish they should swap roles.
6. Ask students to find the corresponding terms in the text.
   Key: point, intersection, ray, parallel
7. Ask students to draw separate pictures consisting of lines and angles. Then ask them to describe their pictures to their partners, asking them to draw the pictures according to instructions. They should then compare the pictures and swap roles.
8. Ask students to put the words in the right order to form correct sentences
   Key: a) A line is one of the basic terms in geometry.
   b) A point is a dot on a piece of paper.
   c) The point that lines share is called the point of intersection.
   d) A line segment does not extend forever, but has two distinct endpoints.
   e) The point where the ray begins is known as its endpoint.
   f) Two rays that share the same endpoint form an angle.
   g) We measure the size of an angle using degrees.
   h) A right angle is an angle measuring 90 degrees.

Chapter 7

1. Ask the students what geometrical shapes they know. Write the terms on the board.
2. Ask students to listen to the words and repeat after the recording.
3. Ask students to read the text carefully and find the words in the text.
4. Ask the students to do the true/false sentences in pairs. Check answers with the whole group.
   Key: a) true, b) false, c) false, d) false, e) false, f) true
5. Ask students to answer the questions according to the text.
   Key: a) a collection of points in a plane that are all the same distance fro a fixed point
   b) bases
   c) rectangle
   d) hypotenuse
   e) legs
   f) it has the sides of the same length and its angles are the same.
6. Ask students to draw pictures containing 15 different geometrical shapes. They should then show their pictures to their partners asking them to find all the figures and name them in English. Swap roles.

Remark! The following additional vocabulary may be needed here:

to the left, to the right, in the corner, above, below, in front of, behind, inside, next to, beside.

7. Ask students to decide what the names are. Then practice with the recording.

   Shapes vocabulary **Key**:
   trapezoid perpendicular line segments diameter parallelogram square
   point obtuse angle circle octagon acute angle

isosceles triangle ray parallelogram vertex quadrilateral
closed curve scalene triangle parallel line segments line segment rectangle
intersecting line segments chord right angle line radius

Chapter 8

1. Books closed. Ask your students what basic solids and space figures they know.

2. Play the recording, let the students repeat the words.

3. Ask the students to read the text carefully and find any other terms

4. Ask the students to answer the questions.
Key: a) 4, b) 1, c) 0, d) radius, e) 2, f) it has depth, width, height.

5. Ask the students to work in groups of three. Each group should prepare 6 true/false sentences about the text. Then they should change groups. In the new groups they should test their partners.

6. Each student should draw a picture containing 15 figures. Then they show the pictures to their partners, asking them to find and name the figures. Then they should swap roles.

7. Ask students to find the plural forms of the following words:

depth – depths	height – heights
sphere – spheres	ball – balls
cross-section – cross-sections	cone – cones
slice – slices	box – boxes
surface – surfaces	radius – radii/radiuses

8. Ask students to put the following words in the right order, to form correct sentences:
Key: a) A prism is a space figure with two congruent, parallel bases.
b) The distance from the center to the surface of the sphere is called its radius.
c) A cube is a three-dimensional figure having six matching square sides.
d) A space figure has depth in addition to width and length.
e) A space figure having all flat faces is called a polyhedron.
f) Volume is a measure of how much space a space figure takes up.

9. Ask students to work in pairs. One of them is A (text in Appendix 2A), the other is B (text in Appendix 2B). They should read their texts and then ask their partner questions about the missing information in the gaps. Walk around the classroom monitor the correctness of the questions. Then check the answers with the whole group.

Chapter 9

1. Warm up (A game explaining what probability is all about)
Ask three students to stand in the corner.
Ask one of the "corner" students: "When is your birthday?" Write the day and month at the blackboard.
What's the probability of the 2^{nd} person sharing the same birthday? KEY: 1/365
What's the probability of the 2^{nd} birthday being different? KEY: 364/365
What's the probability of the 3^{rd} birthday being different? KEY: 363/365

Ignoring the issues of leap years the problem is solved as follows:

Explanation:

When the first person announces their birthday, the probability of the second person sharing the same birthday is 1/365. Conversely, the probability of the second birthday being different is the opposite of the first calculation, 364/365.

2. Ask the students to repeat the words after the recording. Explain their meaning.

3. Ask students to read the text quietly. Ask them what examples of the theory of probability are described in the text.

4. Ask students what examples of application of the theory of probability they can think of (for instance: gambling – cards, horse races, lotteries, bets, predicting other events, assessing risk, etc.)

5. In pairs think of an example of application of the probability theory, describe it and then present to the group.

6. Write 6 True/False sentences based on the text and quiz each other in pairs or groups of three.

7. Find 10 useful terms from this chapter. Quiz your partner. Swap roles.

8. Ask students to read the text carefully explaining the difficult words. When they have finished, ask them to work in groups of three and try to find the solutions.

Here you are given the possible solutions as a key. When they finish discussing their own solutions, you could photocopy these ones and give them to small groups to discuss.

Hermit Model

- Use a six-sided die where each side represents a hermit.
- Roll the die to see which hermit gets the disease.
- Roll the die again to see which hermit is visited and gets the disease.Continue rolling until an immune hermit (one of the numbers that has already been rolled) is visited.
- **NOTE**: If the same number is rolled one after the other, ignore the second roll, since these hermits do not visit themselves.
- Count how many different numbers were rolled (how many hermits got the disease).
- Repeat steps 2–5 many times.
- Average the number of hermits who got infected per outbreak.

There are many other possible ways to model this problem. Can you think of any?
How could you model an island with 4 hermits? 10 hermits? 52 hermits?
Sometimes it can be handy to refer back to the original problem for insight.
(The computer can run these trials much more quickly than any human can. If you have a Macintosh, you can download a handy program for solving this problem).

Once you have run some experiments and come up with some expected values, try your hand at finding an analytical solution.

Analytical Solution

In order to solve this problem analytically, you must find the probability that each possible number of hermits will be infected. There will be at least two infected (one gets the disease and visits another) and at most six infected (all get visited without anyone getting visited twice).

Look at each case separately:

# of infected hermits	explanation	probability
0	This cannot happen, because at least one hermit must get sick in order to begin the outbreak (trial).	**0**
1	This cannot happen, because at least one hermit must get sick and then he visits someone else who, in turn, also gets sick.	**0**
2	Once the second hermit is visited, there are five hermits that he can visit (one of which is immune). If he visits the immune hermit, only two will be infected.	**1/5 = 0.2000**
3	For three hermits to be infected, the second hermit must have visited one of the four non-immune hermits. Then, the third hermit must visit another hermit (two of which are immune). If he visits one of the two immune hermits, only three will be infected.	(4/5)(2/5) = **8/25 = 0.3200**
4	For four hermits to be infected, the second hermit must have visited one of the four non-immune hermits. Then, the third hermit must have visited one of the three remaining non-immune hermits. Then that hermit must visit one of the three immune hermits.	(4/5)(3/5)(3/5) = **36/125 = 0.2880**
5	For five hermits to be infected, the second hermit visited one of four non-immune hermits, the third hermit visited one of three non-immune hermits, the fourth visited one of two non-immune hermits and the fifth visited one of four immune hermits.	(4/5)(3/5)(2/5)(4/5) = **96/625 = 0.1536**
6	For all the hermits to be infected, the second hermit visited one of four non-immune hermits, the third hermit visited one of three non-immune hermits, the fourth visited one of two non-immune hermits, and the fifth visited the last non-immune hermit	(4/5)(3/5)(2/5)(1/5) = **24/625 = 0.0384**

Now check these answers. Remember, the total probability of all possibilities must equal 1.

Total probability = 0 + 0 + (1/5) + (8/25) + (36/125) + (96/625) + (24/625) =
(125/625) + (200/625) + (180/625) + (96/625) + (24/625) = 625/625 = 1.

As you can see, these numbers do check out. Now we find the expected value, E(x), by multiplying each value by its respective probability and adding them all together:

E(x) = 0(0) + 1(0) + 2(1/5) + 3(8/25) + 4(36/125) + 5(96/625) + 6(24/625) =
0 + 0 + (2/5) + (24/25) + (144/125) + (480/625) + (144/625) =
(250/625) + (600/625) + (720/625) + (480/625) + (144/625) =

(2194/625) = 3.5104

Thus, by averaging all the trials from your model, you should get something near 3.5104 as your answer.

Chapter 10

1. Books closed. Ask students to define combinatorics. Accept various ideas.
2. Ask students to repeat the following words after the recording:
   set, factorial, square, collection, subset, combination, permutation, graph, enumeration, design, discrete, complement set, disjoint set, a deck of playing cards, boundary, principal, significant, to concern, retrieval.
3. Ask students to choose five of these terms and discuss what they mean in pairs.
4. Ask students to read the text and find these words in context.
5. Ask students to write the definitions individually, referring to the text. When they finish ask them to make groups of 3 or 4, read the definitions in groups and guess the words.
6. Ask students to cover the text and decide individually if the statements are true or false. When they have decided individually ask them to compare answers in groups of 3 or 4. If students find it difficult to remember allow referring to the text.
   Key: a) false
   b) false
   c) true
   d) true
   e) false
   f) false
7. Ask students to prepare 5 questions and ask them in groups.
8. Ask students to work in pairs and decide who is A and who is B.
   A should read the text in Appendix 4A, B – the text in Appendix 4B. When they have finished ask them to prepare questions about the missing information in their texts. Walk around the classroom monitoring the correctness of the questions. Then they should work in pairs asking their partner the questions to obtain the missing information.

   Key A: 1. heads, 2. covers, 3. arbitrary, 4. final, 5. initial, 6. circle, 7. operation, 8. number, 9. odd, 10. regions, 11. tails, 12. configuration.

   Key B: 1. coin, 2. regions, 3. turns over, 4. condition, 5. no, 6. proceed, 7. initial, 8. heads, 9. subject, 10. circles, 11. triple-region, 12. sum.

Chapter 11

Books closed

1. Ask students to work out the definition of differential equations in pairs. Get feedback with the whole group. If they find it difficult write the following equation on the board:

   $$A\frac{dx}{dt} + Bx(t) = C$$

   And elicit: is it a differential equation? (Answer: it is).

2. Ask students what applications of differential equations they know (physics, chemistry, biology, economics, mathematics).
3. Ask students to repeat the following words after the recording:

   equation, derivative, function, variable, relativity, coefficient, dynamics, ordinary, linear, solution, boundary, value, homogenous.
4. Tell students to read the text and find other useful expressions.
5. Ask students to write 6 true/false sentences individually. Make sure they know the answers themselves. Then ask them to work in groups of three, take turns to read the sentences and decide if they are true or false.
6. Ask students to read the second text and find answers to the questions. All the answers can be easily found in the text. Check with the whole group.

   Key: a) The **order** of the differential equation is the order of the highest derivative of the unknown function involved in the equation.

   b) A linear differential equation of order n is a differential equation written in the following form:

   $$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_1(x)\frac{dy}{dx}+a_0(x)y=f(x)$$

   c) the only operations which are accepted for the variable y are:

   d) differentiating y;

   e) multiplying y and its derivatives by a function of the variable x;

   f) adding what you obtained in (ii) and let it be equal to a function of x.

   g) a problem in which we are looking for the unknown function of a differential equation where the values of the unknown function and its derivatives at some point are known is called an **initial value problem** (in short IVP).

   h) if no initial conditions are given, we call the description of all solutions to the differential equation the **general solution**.
7. Ask students to cover the text, but leave the questions uncovered. They should ask each other the questions and answer without looking back at the text.
8. Ask students to read the equations in pairs.
9. Ask students to work in pairs. Person A – should work with the text in Appendix 3A, person B – the text in Appendix 3B. They should read their text carefully and prepare questions about the missing information in the gaps. Then they should ask their questions in pairs trying to fill the gaps. When they finish ask them to check their text with their partner's.

   Key A: 1. students, 2. path, 3. jagged, 4."limit", 5. itself, 6. calculus, 7. numbers, 8. the number 1.0, 9. size, 10. 9

   Key B: 1. calculus, 2. equilateral, 3. ADEFC, 4. zero, 5. length, 6. extensively, 7. the fallacy, 8. an integer, 9. note, 10. digits

Chapter 12

1. Ask students about their definition of abstract algebra.
2. Play the recording, ask students to repeat the words.
3. Ask students to read the text carefully, tell them to write down any new terms.

4. Ask students to answer the questions according to the text.

Key:
1. It deals with groups, rings, fields, modules, vector spaces. Algebras.
2. Beginning of the 20th century,
3. Modern algebra
4. Due to the need for more intellectual rigor in mathematics.
5. As axiomatic systems.
6. It is used to distinguish the aggregate of high school algebra from elementary algebra.
7. From other fields of mathematics.
8. Quasigroups, monoids, semigroups, rings and fields, modules and vector spaces, algebras over fields…)
9. They are the same.
10. Functions under composition and matrices under multiplication.

5. Group activity:

BRING DICTIONARIES TO THE CLASROOM

Tell the students which number they are from 1 to 4 (1, 2, 3, 4). Ask 1s, 2s, 3s, 4s to sit together in groups. Assign text 1 for group1, text 2 for group 2, etc.

[This is best done if you photocopy the texts and cut them separately, so that each group gets only one text.]

Ask students to read their texts carefully, find new terms and explain their meanings using a dictionary.

They should prepare a summary of the text, so that they can talk about the content to students from other groups. Allow 15 minutes for this.

When they are ready rearrange groups. 1, 2, 3, 4 should sit together as one group now. In those new groups they should present the content of their texts. The others should take notes. When they have finished ask them to read all the texts.

6. Ask students to find 5 difficult words from this lesson. Tell them to work in pairs and prepare their definitions. They should then join another pair and in groups of 4 read their definitions. The other pair should guess the words. Swap roles.

7. Ask students to work in pairs and prepare 10 True/False sentences to the four texts. When they have finished ask them to join another pair and make them decide whether their sentences are true or false. Tell them to swap roles.

Chapter 13

1. Ask students to choose 10 expressions from the previous chapter (Abstract Algebra Part I) and quiz each other in pairs to revise the definitions.

2. Ask students to repeat the terms after the recording.

3. Ask students to read the text and write down any new terms.

4. Ask students to prepare 5 questions about the text. Make them work in groups of three, asking and answering their questions in turns.

5. Ask students to make lists of important terms from the chapter. Make them work in pairs. When they have finished ask them to compare their lists with another pair.

6. Ask students to rearrange the words to form correct sentences:

Key: a) The neutral element is usually called the identity element.
b) A group is called finite if it has finitely many elements.
c) Different operations on the same set define different groups.
d) A ring is an algebraic structure in which addition and multiplication re defined.
e) The branch of abstract algebra which studies rings is called ring theory.
f) An element *a* in a ring is called a unit if it is invertible with respect to multiplication.
g) The requirement $0 \neq 1$ ensures that the set which only contains a single element is not a field.
h) The multiplicative inverse of a product is equal to the product of the inverses.

7. Ask students to fill in the missing nouns and verbs in the chart:

Key:

NOUN	VERB
Operation	OPERATE
SATISFACTION	Satisfy
REPRESENTATION	Represent
Application	APPLY
ASSOCIATION/ASSOCIATIVITY	Associate
Definition	DEFINE
COMPOSITION	Compose
Multiplication	MULTIPLY

8. Ask students to find a factual mistake in each of the given sentences.

Key: a) The neutral element is usually called the identity element for a…
b) A group is called finite if it has finitely many elements…
c) Usually the operation is thought of as an analogue of multiplication…
d) In mathematics, a ring is an algebraic structure in which …
e) The branch of abstract algebra which studies rings is called ring theory.
f) A ring is a set of two binary operations, called addition and multiplication…
g) An element in a ring is called a unit if it is invertible…
h) The requirement (…) ensures that the set which only contains a single element is not a field.

Additional Material

1. As an integral part of the course the authors suggest that students should prepare presentations in small groups. Materials can be found in the library of the Gdańsk University of Technology.

www.bg.edu.pl (bazy danych, czasopisma online)

2. If you are left with some free time at the end of a lesson you can do some of the following exercises (this can be done as a competition against time when students are divided into smaller groups. The group that is the first one to collect all the answers, wins).

Answers: 1. Gauss and Pascal, 2. Gauss, 3. Poincare, 4. Stefan Banach, 5. Cardano, 6. Pascal, 7. Leibniz, 8. Leibniz, 9. l'Hopital, 10. Newton, 11. Ferrari, 12. l'Hopital, 13. Poincare, 14. Leibniz, 15. Euler, 16. Fermat.

3. Ask students to work in groups of four. Each of them reads one biography. They should then work for a few minutes preparing arguments for "their" mathematician being the greatest. Then in their small groups they have to try to convince their opponents. This can also be done as a whole class discussion.
4. Ask students to work in groups of two or three and prepare questions about the mathematicians. Ask them to join another group and quiz each other